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On homogenized conductivity and fractal structure in a high contrast continuum percolation model

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ABSTRACT

In the previous article (Matsutani et al., 2012) we numerically investigated an electric potential problem with high contrast local conductivities (γ_0 and γ_1, $0 < \gamma_0 \ll \gamma_1$) for a two-dimensional continuum percolation model (CPM). As numerical results, we showed there that the equipotential curves exhibit the fractal structure around the threshold p_c and gave an approximated curve representing a relation between the homogenized conductivity and the volume fraction p over $[p_c, 1]$. In this article, using the duality of the conductivities and the quasi-harmonic properties, we re-investigate these topics to improve these results. We show that at $\gamma_0 \to 0$, the quasi-harmonic potential problem in CPM is quasiconformally equivalent to a random slit problem, which leads us to an observation between the conformal property and the fractal structure at the threshold. Further we extend the domain $[p_c, 1]$ of the approximated curve to $[0, 1]$ based on the these results, which is partially generalized to three dimensional case. These curves represent well the numerical results of the conductivities.

1. Introduction

In the series of articles [1–3], we numerically investigated the electric potential problem with high contrast local conductivities (γ_0 and γ_1, $0 < \gamma_0 \ll \gamma_1$) for continuum percolation models (CPMs) in order to reveal the electrical properties of a real material consisting of conductive nano-particles in an insulator. We solved generalized Laplace equations with mixed boundary conditions using the finite difference method. In Ref. [3], we investigated two-dimensional case and numerically showed that the equipotential curves exhibit the fractal structure around the threshold p_c and gave an approximated curve representing a relation between the homogenized conductivity and the volume fraction p over $p \in [p_c, 1]$, which we call *conductivity curve*. The fractal structure in Ref. [3] shows the electrical properties of the system, i.e., there appear quasi-equipotential clusters there, in which the potential distribution is constant or nearly constant. On the other hands, due to the ultra high conductivity of an insulator, we have a non-vanished effective conductivity for a smaller volume-fraction p than the threshold p_c. It is also important to determine the dependence of the homogenized conductivity on volume fraction p for every $p \in [0, 1]$.

In this article, we go on to investigate the electric potential distribution on a two-dimensional continuum percolation model (CPM) [4] with a high contrast local conductivity to improve the results in Ref. [3] from more mathematical viewpoints. We consider the simplest Boolean model of CPM, in which disks of unit radius are centered at the points of the

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Poisson point process with density λ [4,5]. In order to consider the duality between the occupancy state (OS) and the vacancy state (VS), we handle both types in this article; the OS-type is set so that occupied regions have the local conductivity $\gamma_1 = 1$ and the vacant regions have infinitesimal one γ_0, whereas in the VS-type, the occupied regions have γ_0 and the vacant regions have γ_1. VS-type is sometimes called the Swiss cheese model in physics literature if $\gamma_0 = 0$.

Using the duality of two dimensional case, the purposes of this article are to find the geometric properties of the potential distribution, especially the fractal structure of the equipotential curves around the threshold, and to extend the domain $[p_c, 1]$ of conductivity curve for the homogenized conductivity to $[0, 1]$, which leads us a novel parameterization of the conductivity curve even of three dimensional case.

The homogenized conductivity in CPM has been rigorously studied well and has a long history as recently Kontogiannis [6] gave a nicer review of these studies and an extension. The studies from more practical viewpoints based on the numerical results are in Refs. [7,8]. In this article, based on these rigorous results and numerical computations, we give an improvement of the previous arguments in Ref. [3]. Thus the conventions and notations in this article completely differ from the previous ones [1–3] since they were written in the framework of physics. In general, the homogenization is assumed a periodicity of its associated system. Instead of the periodicity, we use the ergodicity of CPM to homogenize the conductivity following Refs. [9,10] because the Boolean model has the ergodic properties [6,10].

Due to the duality of OS- and VS-types, it is well-known that our second order partial differential equations (PDEs) have the quasiconformal properties of the PDEs as the quasi-harmonic system [11]. Using the quasiconformal property and the Keller–Dykhne reciprocity law [12,13] of the conductivity, the equipotential curves are represented mathematically. We showed that in Theorem 4.1 for the infinitesimal limit of γ_0, the OS-type potential distribution in CPM forms a quasiconformal map from the domain to $[0, 1] \times [0, 1]$ with random slits. It implies that at $\gamma_0 \to 0$, the quasi-harmonic potential problem in CPM is quasiconformally equivalent to a random slit problem. Since one of our purposes is to reveal the fractal structure of the equipotential curves [3], we give Theorem 4.2 and using it, show an observation in Remark 4.2 that the equipotential curves at $p \nearrow p_c$ are conformally mapped to the family of the lines with infinite length under some assumptions. Further the conformal map gives a relation to a Riemann sphere $P\mathbb{C}^1$.

The other purpose is to find the properties of the conductivity under the threshold. In order to use the duality, we give our numerical computations of the conductivity of VS-type using the Monte-Carlo method and finite difference method as we did for OS-type in Ref. [3]. Then based upon these results, we give novel approximation formulae of the conductivity curves. In other words, using the duality, we give approximation formulae of the conductivity curves in Remark 6.1 and Fig. 8(a). Fig. 8(a) shows that the approximation formulae represent well the computational results of the homogenized conductivities.

It leads us another approximation formula of the conductivity curve of three-dimensional case as in Remark 6.2 and Fig. 8(b). In Refs. [1,2], we have numerically studied the homogenized conductivity on the CPM with OS-type local conductivity mainly in three dimensional case by solving the generalized Laplace equation with a certain Dirichlet–Neumann boundary conditions. The proposed formula also approximates well the conductivity curve of the random spheres in Remark 6.2 as in Fig. 8(b). It means that in terms of the novel formula, we can parameterize the conductivity properties discussed in Refs. [1,2].

Contents in this article are as follows. Section 2 gives our model and mathematical preliminaries. Section 2.1 is an review on our Boolean model of CPM based upon the Poisson point process, in which we introduce OS- and VS-types, and Section 2.2 gives our PDEs or the generalized Laplace equations on the CPMs. Section 3 also shows the well-established results homogenization of the conductivity associated with our PDEs. In Section 4, after we review the quasiconformal properties of the PDEs as the quasi-harmonic system, we show that in the infinitesimal limit of γ_0, the potential distribution is quasiconformally equivalent to a configuration of a random slit model in Theorem 4.1, which is the first our main result. The conformal structure leads us Theorem 4.2 and a novel observation on the fractal structure in Remark 4.2 that the equipotential curves which is the second our main result. In Section 5 we give our numerical computations using the Monte-Carlo method. Based upon these results, in Section 6 we give novel approximation formulae of the conductivity curves in Remark 6.1 and their extension to three-dimensional case in Remark 6.2, which are the third our main result. Fig. 8 shows that they represent the computational results well.

2. Models and preliminary

2.1. Model, definitions and notations

Let $\hat{\mathcal{Q}}(\mathbb{C})$ be the set of all countable subsets $\hat{X}$ of the region $\mathbb{C} \equiv \mathbb{R}^2$ satisfying $N_K(\hat{X}) < \infty$ for every compact subset $K \subset \mathbb{C}$, where $N_K(\hat{X})$ is the number of the points of $\hat{X} \cap K$. Thus $\hat{\mathcal{Q}}(\mathbb{C})$ has the non-negative valued Radon measure and is equipped with σ-field $\mathfrak{B}(\hat{\mathcal{Q}}(\mathbb{C}))$. Let ℓ be the Lebesgue measure of $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Then we consider the Poisson point process $(\hat{\mathcal{Q}}(\mathbb{C}), \mathfrak{B}(\hat{\mathcal{Q}}(\mathbb{C})), \hat{\mathbf{P}}_\lambda)$, i.e., for any disjoint region $\{A_1, A_2, \ldots, A_m\} \subset \mathfrak{B}(\mathbb{C})$ such that $\ell(A_i) < \infty$ $i = 1, 2, \ldots, m$, $N_{A_1}(\hat{X})$, …, $N_{A_m}(\hat{X})$ are independent random variables on the probability space $(\hat{\mathcal{Q}}(\mathbb{C}), \mathfrak{B}(\hat{\mathcal{Q}}(\mathbb{C})), \hat{\mathbf{P}}_\lambda)$ and for $n \in \mathbb{N}_0$,

$$\hat{\mathbf{P}}_\lambda(N_{A_i} = n) = \frac{(\lambda\ell(A_i))^n}{n!} \exp((\lambda\ell(A_i))), \quad i = 1, 2, \ldots, m, \quad n \in \mathbb{N}_0.$$