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Identification of piecewise constant sources in non-homogeneous media based on boundary measurements

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ABSTRACT

In this study, we consider the inverse problem of determining a source term in an elliptic problem based on boundary measurements. The boundary measurements allow the unique determination of the harmonic component of the source, and thus a priori information is required for its complete identification. Using this a priori information, we determine the compactness of the class of sources and a uniqueness theorem for its identification from Cauchy data, thereby allowing us to propose a stable algorithm for finding "approximate solutions" of the inverse problem. The proposed procedure is demonstrated by individual cases, where we know the "geometry" of the inner region and the source takes a constant value.

1. Introduction

In this study, we consider a problem that corresponds to the physical situation of a non-homogeneous conductive medium Ω immersed in a perfect insulator. For simplicity, but without loss of generality in the field of applications, we assume that Ω is a simply connected open set in $\mathbb{R}^N, \Omega_1$ is another open simply connected set compactly contained in Ω, and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. We denote S_1 and S_2 as the boundary of Ω_1 and the exterior boundary of Ω_2, respectively. We assume that Ω_1 and Ω_2 have different conductivities of σ_1 and σ_2, respectively. We denote u_1 and u_2 as the electric potentials generated in Ω_1 and Ω_2, respectively. Thus, the simplest stationary model that describes the behavior of u_1 and u_2 is given by:

$$-\sigma_1 \Delta u_1 = f, \quad \text{in} \quad \Omega_1, \tag{1}$$

$$\Delta u_2 = 0, \quad \text{in} \quad \Omega_2, \tag{2}$$

$$u_1 = u_2, \quad \text{on} \quad S_1, \tag{3}$$

$$\sigma_1 \frac{\partial u_1}{\partial n_1} = \sigma_2 \frac{\partial u_2}{\partial n_1}, \quad \text{on} \quad S_1, \tag{4}$$

$$\sigma_2 \frac{\partial u_2}{\partial n_2} = 0, \quad \text{on} \quad S_2, \tag{5}$$

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where $\frac{\partial u_i}{\partial n_j}$ denotes the normal derivative of u_i in S_j with respect to the normal unitary vector n_j, which is exterior to $\Omega_j, i, j = 1, 2$.

In Section 2, we present a summary of the solubility of the problem (1)–(5). Further details can be found in [1,2].

The problem addressed in the current study corresponds to the inverse problem of source identification in the problem (1)–(5) using additional data, $u_2|_{S_2} = \phi$. Using a similar approach in specific situations with relevant a priori information on f and other additional data associated with u, problems of this type have been studied in [3–5,1,6–14]. In the following, we refer to the boundary value problem (1)–(5) as EBP.

In this study, we assume that there is only one harmonic source $h \in L_2(\Omega_1)$ and it is orthogonal to the constants in the usual scalar product of $L_2(\Omega_1)$, i.e., $\langle h, 1\rangle_{L_2(\Omega_1)} = \int_{\Omega_1} h = 0$, which reproduces the measurement ϕ on S_2. In the general case where a priori information is used, it is assumed that $f \in \mathcal{F}$, where $\mathcal{F}$ is a class of functions in $L_2(\Omega_1)$ with certain special properties, and thus a good criteria for choosing a "first approximation" of the source in $\mathcal{F}$ that "better recovers" the measurement ϕ is obtained by solving the optimization problem:

$$\min_{f\in\mathcal{F}} \|f - h\|^2_{L_2(\Omega_1)} \rightsquigarrow f^*. \tag{6}$$

In Section 3, we demonstrate the important role played by f^* in solving the identification problem by minimizing the fitting functional

$$\min_{f\in\mathcal{F}} \|u_{2f}|_{S_2} - \phi\|^2_{L_2(S_2)}, \tag{7}$$

where u_{2f} represents the solution in Ω_2 of the EBP for f and $u_{2f}|_{S_2}$ denotes the trace of u_{2f} on S_2.

Unlike (6), we note that the minimization of the functional (7) requires the solution of the forward problem associated with EBP in each iteration step, which can be computationally expensive.

2. Weak solution of EBP and the associated inverse problem

Let $L_2(\Omega_i), L_2(S_i), i = 1, 2$ and $L_2(\Omega)$ be the spaces of square integrable functions defined on Ω_i, S_i and Ω, respectively. We denote $H_1(\Omega_i), i = 1, 2$ and $H_1(\Omega)$ as the corresponding Sobolev spaces of the functions in $L_2(\Omega_i)$ and $L_2(\Omega)$, respectively, the generalized first derivatives of which are also square integrable functions. We denote $H_{\frac{1}{2}}(S_i), i = 1, 2$ as the subspaces of $L_2(S_i)$, which comprise the traces to S_i of the functions in $H_1(\Omega_i)$. Finally, $H(\Omega_i), i = 1, 2$ denote the subspaces (closed) of harmonic functions in $L_2(\Omega_i)$. We use the superscript (1) in these spaces to denote the spaces of functions that are orthogonal to the constants with respect to the corresponding scalar product, i.e., if W is any previous space then, $W^{(1)} = \{w \in W : \langle w, 1\rangle_W = 0\}$, where $\langle , \rangle_W$ is the scalar product of W. We also use the superscript $\perp$ to denote the orthogonal subspace to a subset of square integrable functions.

Definition 2.1. Given $f \in L_2(\Omega_1)$, a function $u \in H_1(\Omega)$ is a weak solution of EBP if it satisfies the following relationship

$$\int_{\Omega_1} f v_1 d\Omega_1 = \int_{\Omega_1} \sigma_1 \nabla u_1 \cdot \nabla v_1 d\Omega_1 + \int_{\Omega_2} \sigma_2 \nabla u_2 \cdot \nabla v_2 d\Omega_2, \tag{8}$$

$\forall v \in H_1(\Omega)$, where $v_i = v|_{\Omega_i}, i = 1, 2$ (see [1,2]).

It is easy to see that the relation (8) holds for any classical solution of the problem (1)–(5). Indeed, it is sufficient to transform the expression

$$\int_{\Omega_1} \sigma_1 \nabla u_1 \cdot \nabla v_1 d\Omega_1 + \int_{\Omega_2} \sigma_2 \nabla u_2 \cdot \nabla v_2 d\Omega_2 \text{ with } v \in H_1(\Omega),$$

using the Green formula to obtain (8).

The following result is known.

Theorem 2.2. *The weak solution of the EBP exists if and only if $f \in L_2^{(1)}(\Omega_1)$. In this case, the weak solution is unique in $H_1^{(1)}(\Omega_1)$ and the following estimate holds*

$$\|u\|_{H_1(\Omega)} \leqslant C\|f\|_{L_2(\Omega_1)}, \tag{9}$$

where C is a constant that does not depend on f (see Section 1.2 of Chapter IV [2]).

From inequality (9), it follows that the operator T, which for each $f \in L_2^{(1)}(\Omega_1)$ is associated with the weak solution $u \in H^{(1)}(\Omega_1)$ of the problem (1)–(5), is continuous from $L_2^{(1)}(\Omega_1)$ to $H_1(\Omega)$. Since the trace operator is compact from $H_1(\Omega)$ to $L_2(S_2)$, then it is concluded that the composed operator $\mathcal{A}_o : L_2^{(1)}(\Omega_1) \to L_2(S_2)$, defined by $\mathcal{A}_o(f) = u_2|_{S_2}$, is compact.

The following result can be found in [1].