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Solution of Cauchy type singular integral equations of the first kind by using differential transform method

M. Abdulkawi

Department of Mathematics, Faculty of Science and Arts, Najran University, Saudi Arabia

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ABSTRACT

The differential transform method is extended to solve the Cauchy type singular integral equations (CSIEs) over a finite interval. New theorems for transformation of Cauchy singular integrals are given with proofs. Approximate solutions of CSIEs with two types of kernels, Degenerate and convolution, are obtained. The system of linear equations for characteristic equation is solved analytically. Numerical results are shown to illustrate the efficiency and accuracy of the present method.

1. Introduction

Consider the Cauchy type singular integral equations (CSIEs) of the form

$$\int_{-1}^{1} \frac{\psi(t)}{t-x}\,dt + \int_{-1}^{1} K(x,t)\psi(t)\,dt = f(x), \quad -1 < x < 1, \tag{1}$$

where $K(x,t)$ and $f(x)$ are given real valued functions belonging to the Hölder class and $\varphi(t)$ is to be determined. Eq. (1) arises in many diverse areas of physical sciences like neutron transport, contact problems in elasticity and wave guide theory [1–3]. The integral in Eq. (1) is considered as Cauchy principal value integral. If $K(x,t) = 0$ in Eq. (1) then we obtain the characteristic singular integral equation

$$\int_{-1}^{1} \frac{\psi(t)}{t-x}\,dt = f(x), \quad -1 < x < 1. \tag{2}$$

It is known that the analytical solution of Eq. (2) is given by the following expression [4]

Case (I): The solution is bounded at the end $x = -1$, but unbounded at the end $x = 1$,

$$\psi(x) = -\frac{1}{\pi^2}\sqrt{\frac{1+x}{1-x}}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}\frac{f(t)}{t-x}\,dt. \tag{3}$$

Case (II): The solution is bounded at the end $x = 1$, but unbounded at the end $x = -1$,

$$\psi(x) = -\frac{1}{\pi^2}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{f(t)}{t-x}\,dt. \tag{4}$$

E-mail address: kawi2011@gmail.com

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By solving Eq. (1) with respect to its characteristic part, we find that it is equivalent to the Fredholm equation type of the second kind [2]

Case (I):

$$\left.\begin{array}{l}\psi(t)+\int_{-1}^{1} N_1(t,\tau)\psi(\tau)\,d\tau = F_1(t),\\ N_1(t,\tau)=\frac{1}{\pi^2}\sqrt{\frac{1+t}{1-t}}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{K(x,\tau)}{t-x}\,dx,\\ F_1(t)=\frac{1}{\pi^2}\sqrt{\frac{1+t}{1-t}}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{f(x)}{t-x}\,dx.\end{array}\right\} \tag{5}$$

Case (II):

$$\left.\begin{array}{l}\psi(t)+\int_{-1}^{1} N_2(t,\tau)\psi(\tau)\,d\tau = F_2(t),\\ N_2(t,\tau)=\frac{1}{\pi^2}\sqrt{\frac{1-t}{1+t}}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{K(x,\tau)}{t-x}\,dx,\\ F_2(t)=\frac{1}{\pi^2}\sqrt{\frac{1-t}{1+t}}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{f(x)}{t-x}\,dx.\end{array}\right\} \tag{6}$$

Kim [5] investigated the problem of finding the bounded solution of Eq. (2). He used the method biased on Gaussian quadrature with choosing the zeros of first and second kind Chebyshev polynomials as the collocation and abscissae points respectively. Chakrabarti and Berge [4] have proposed an approximate method to solve Eq. (1) using polynomial approximation of degree n for four cases. Eshkovatov et al. [6] have presented the efficient approximate method for solving Eq. (2) using Chebyshev polynomials of the first, second, third and fourth kinds with corresponding weight functions for four cases. Nik Long et al. [7] presented semi-bounded numerical solutions for Eq. (1). They used the truncated Chebyshev series of the third and fourth kinds with the corresponding weight functions. Mohankumar and Natarajan [8] discussed a numerical solution for a typical Cauchy singular integral equation. They showed a solution prescription that combines a polynomial expansion for the unknown, a collocation procedure for fixing the expansion coefficients and a double exponential quadrature for the Cauchy principal value integral.

In this paper we present approximate solution technique for solving Eqs. (1) and (2) in two cases:

Case (I): The solution is bounded at the end $x=-1$, but unbounded at the end $x=1$.

Case (II): The solution is bounded at the end $x=1$, but unbounded at the end $x=-1$.

2. Differential transform method

The Transformation of the nth derivative of a function f in one variable is as follows [9,10]

$$F(k)=\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{7}$$

and the inverse transformation is defined as

$$f(x)=\sum_{k=0}^{\infty}F(k)\,(x-x_0)^k. \tag{8}$$

Theorem 1. *If $f(x)=x^n$, then $F(k)=\delta(k-n)$, where*

$$\delta(k-n)=\begin{cases}1, & k=n,\\ 0, & k\neq n.\end{cases}$$

Theorem 2. *If $f(x)=ag(x)$, then $F(k)=aG(k)$, where a is a constant and $G(k)$ is a differential transform of $g(x)$.*

Theorems 1 and 2 can be deduced from Eq. (7) with assuming that $x_0=0$.

Theorem 3. *If $g(x)=\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{\varphi(t)}{t-x}\,dt$, then the differential transform of g is*

$$G(\bar{k})=\sum_{k=1}^{N}\sum_{k_1=0}^{k-1}\Phi(k)[h(k-k_1-1)+h(k-k_1)]\,\delta(k_1-\bar{k})+\pi\Phi(\bar{k}),\quad N\to\infty.$$

where $\Phi(k)$ is the differential transform of φ and the constants $h(m)$ are defined as

$$h(m)=\begin{cases}\pi, & m=0,\\ 0, & m \text{ is odd},\\ \frac{1\cdot3\cdot5\ldots(m-1)}{2^{\frac{m}{2}}\left(\frac{m}{2}\right)!}\pi, & m \text{ is even}.\end{cases} \tag{9}$$