A globally and quadratically convergent smoothing Newton method for solving second-order cone optimization

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A R T I C L E   I N F O

A B S T R A C T

Second-order cone optimization (denoted by SOCO) is a class of convex optimization problems and it contains the linear optimization problem, convex quadratic optimization problem and quadratically constrained convex quadratic optimization problem as special cases. In this paper, we propose a new smoothing Newton method for solving the SOCO based on a non-symmetrically perturbed smoothing Fischer–Burmeister function. At each iteration, a system of linear equations is solved only approximately by using the inexact Newton method. It is shown that any accumulation point of the iteration sequence generated by the proposed algorithm is a solution of the SOCO. Furthermore, we prove that the generated sequence is bounded and hence it has at least one accumulation point. Under the assumption of nonsingularity, we establish the local quadratic convergence of the proposed algorithm without strict complementarity condition. Numerical experiments indicate that our method is effective.

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1. Introduction

The second-order cone (SOC) in $\mathcal{R}^n$, also called the Lorentz cone or the ice-cream cone, is defined as

$$\mathcal{K}^n := \left\{ (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathcal{R}^n : x_1^2 \geqslant \sum_{j=2}^{n} x_j^2, \ x_1 \geqslant 0 \right\},$$

where $n \geqslant 2$ is some natural number. Then the interior of $\mathcal{K}^n$ can be defined by

$$\mathrm{int}\mathcal{K}^n := \left\{ (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathcal{R}^n : x_1^2 > \sum_{j=2}^{n} x_j^2, \ x_1 > 0 \right\}.$$

Second-order cone optimization (SOCO) problem is a class of convex optimization problems in which a linear function is minimized over the intersection of an affine linear manifold with the Cartesian product of second-order cones. In this paper we consider the SOCO in standard format

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$$(P) \quad \min \left\{ c^\mathsf{T} x : Ax = b, \ x \in \mathcal{K} \right\}, \tag{1.1}$$

and the dual problem of (P) is given by

$$(D) \quad \max \left\{ b^\mathsf{T} y : A^\mathsf{T} y + s = c, \ s \in \mathcal{K} \right\}, \tag{1.2}$$

where $A \in \mathcal{R}^{m \times n}$, $c \in \mathcal{R}^n$ and $b \in \mathcal{R}^m$, and $\mathcal{K} \subset \mathcal{R}^n$ is the Cartesian product of second-order cones, i.e.,

$$\mathcal{K} = \mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_r}$$

with $\mathcal{K}^{n_i} \subset \mathcal{R}^{n_i}$ for each i, $i = 1, 2, \ldots, r$, and $n = \sum_{i=1}^{r} n_i$. In the subsequent analysis, we focus our analysis on the case $\mathcal{K} = \mathcal{K}^n$ for simplicity. Our analysis can be easily extended to general cases.

Define

$$\mathcal{F} := \{(x, y, s) : Ax = b, \quad A^\mathsf{T} y + s = c, \ x, s \in \mathcal{K}\},$$

$$\mathcal{F}^0 := \{(x, y, s) : Ax = b, \quad A^\mathsf{T} y + s = c, \ x, s \in \operatorname{int}\mathcal{K}\},$$

respectively. Throughout the paper, we make the following assumptions.

Assumption 1.1. Both (P) and (D) are strictly feasible, i.e., $\mathcal{F}^0 \neq \emptyset$.

Assumption 1.2. A has full row rank.

Under Assumption 1.1, it is well-known that both (P) and (D) have optimal solutions and their optimal values coincide [1], and the SOCO is equivalent to its *optimality conditions*:

$$Ax = b,$$

$$A^\mathsf{T} y + s = c, \tag{1.3}$$

$$x \circ s = 0, \quad x, s \in \mathcal{K}, \ y \in \mathcal{R}^m,$$

where "$\circ$" denotes the Jordan product, which will be presented in the next section.

In the last few years, the SOCO has received considerable attention from researchers because of its wide range of applications. We refer the interested reader to the survey paper by Lobo et al. [2] and the references therein. Many researchers have studied interior-point methods (IPMs) for solving the SOCO and achieved plentiful and beautiful results (see, e.g., [3,4] and the references therein).

Recently, smoothing-type methods have attracted a lot of attention partially due to their encouraging convergent properties and superior numerical performances (e.g., [5–19]). In particular, the smoothing Newton method proposed by Qi et al. [18] has received considerable attention from researchers for its simplicity and weaker assumptions imposed on smoothing functions. The Qi–Sun–Zhou method [18] needs to solve only one system of linear equations and to perform only one line search at each iteration, and it is locally superlinearly/quadratically convergent without strict complementarity. It should be noted that in order to obtain the local superlinear (quadratic) convergence some algorithms (e.g., [5,20]) depend strongly on the assumptions of uniform nonsingularity and strict complementarity conditions. By modifying and extending the Qi-Sun-Zhou method [18], some smoothing Newton methods have been proposed for solving the SOCO (e.g., [6–8,10,19]). These methods reformulate the system (1.3) as a family of parameterized smooth equations and solve the smooth equations approximately by using Newton's method at each iteration. By driving the parameter to converge to zero, one can expect to find a solution of the SOCO.

Motivated by their work, in this paper we propose a new smoothing Newton method for solving the SOCO. Under mild assumptions, we prove that the proposed method is globally and locally quadratically convergent. To compare with existing smoothing methods for the SOCO (e.g., [6–8,10,19]), our method has the following special properties.

(a) It is based on a non-symmetrically perturbed smoothing function, while existing smoothing methods (e.g., [6–8,10,19]) were all designed by some symmetrically perturbed smoothing functions.
(b) In our method, a system of linear equations is solved only approximately by using the inexact Newton method (see, Remark 4.1 below and [21]). Notice that in existing smoothing methods (e.g., [6–8,10,19]), the corresponding system is exactly solved at each iteration.
(c) If Assumptions 1.1 and 1.2 hold, then the iteration sequence generated by our method is bounded and hence it has at least one accumulation point. This result is stronger than the corresponding results in [6–8,10,19].