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Bifurcation, invariant curve and hybrid control in a discrete-time predator–prey system

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ABSTRACT

In this study, complex dynamics of a classical discrete-time predator–prey system are investigated. Rigorous results on the existence and stability of fixed points of this system are derived. It can also be shown that the system undergoes flip bifurcation, Neimark–Sacker bifurcation and codimension-two bifurcation associated with 1:2 resonance using the ideas of center manifold theorem, bifurcation theory and the normal form method. Specially, we give the explicit approximate expression of the invariant curve which is caused by the Neimark–Sacker bifurcation. At the same time, bifurcation phenomena and chaotic features are justified numerically via computing Lyapunov exponent spectrum. Results of numerical simulation verify our theoretical analysis. Finally, we extend the hybrid control strategy (state feed back and parameter perturbation) to control flip bifurcation and Neimark–Sacker bifurcation in two-dimensional discrete system.

1. Introduction

The discrete (and continuous) predator–prey system is a model to describe the population dynamics of two interacting species, a predator and its prey. The complex dynamics of these systems have been hot topics in theoretical and mathematical biology over the past decade (discrete cases [1–18] and continuous cases [19–24]). As opposed to the continuous case, the discrete-time predator–prey system may be more appropriate when populations have non-overlapping generations (difference equations) [4,5]. What's more, the choice of discrete case is also crucial because it may exhibit more complicated dynamic behavior (such as bifurcation, chaos, etc.) and can also provide more efficient computational model for numerical simulations. For example, the famous logistic difference equation $p(n+1) = (1+r)p(n) - rp^2(n)$ is more complicated than the corresponding continuous model $\frac{dp(t)}{dt} = rp(t)(1-p(t))$ [4]. As early as 1970s, May pointed out that a simple discrete system may have complex dynamics [25]. Whether it is from a mathematical point of view or standpoint of ecology, to explore the complex dynamics of discrete systems is very meaningful. In a variety of discrete ecological systems, discrete predator–prey systems are the important class of ecological systems, which have been studied extensively from different perspectives. Discretization of continuous systems is an important way to obtain discrete models. Refs. [3–7,9,12,13,15] discussed some discrete predator–prey systems which are derived from the corresponding continuous systems by the forward Euler scheme. These papers demonstrated that the complex dynamics in these discrete-time predator–prey models take place. In this paper, we consider a classical discrete-time predator–prey system as follows [1,2]:

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$$\begin{cases} x_{n+1} = x_n + rx_n(1 - x_n) - ax_ny_n, \\ y_{n+1} = y_n + ay_n(x_n - y_n). \end{cases} \tag{1.1}$$

where x_n and y_n stand for the densities of prey and predator populations at time n, respectively. The term $x_n + rx_n(1 - x_n)$ represents the rate of the increase of the prey populations in the absence of predator. The term ax_ny_n stands for the rate of decrease due to predation, where the parameter a is the predation parameter. The term $y_n + ay_n(x_n - y_n)$ represents the variation of predator density with respect to the prey population. r and a are positive constants. Although there are many papers dealing with discrete predator–prey systems [1–10,12,15,13,18], the system (1.1) is neither as same as the existing related models nor included by them. Notice that if the predator density disappears in the system (1.1), then the system (1.1) degenerates into the discrete logistic-type model $y_{n+1} = y_n - ay_n^2$. Celik and Duman [1] studied the local stability of the model (1.1), and showed the bifurcation phenomena by numerical simulations. These analyses are far from completion. Thus, we mathematically prove the codimension one (or two) bifurcation of fixed points, including the flip (Neimark–Sacker, abbr. N–S and 1:2 resonance) bifurcation. At the same time, we get the direction of the N–S bifurcation and explicit approximation expression of the invariant curve of N–S bifurcation. Finally, controlling bifurcation is done using hybrid control strategy. Most of these are not given in previous references.

The rest of this paper is organized as follows. In Section 2, existence and stability of fixed points are analyzed. In Section 3, we give some details about bifurcation analysis of codimension-one and codimension-two. In Section 4, accurate control of bifurcation phenomena are described. Finally, some conclusions close the paper in Section 5.

2. Existence and stability of fixed points

It is clear that the fixed points of the system (1.1) satisfy the following equations:

$$\begin{cases} x_n = x_n + rx_n(1 - x_n) - ax_ny_n, \\ y_n = y_n + ay_n(x_n - y_n). \end{cases} \tag{2.1}$$

For a discrete dynamical system on $\mathbb{R}^2$, let the Jacobian matrix of this system evaluated at a fixed point (x, y) be $J|_{(x,y)}$. Assume that λ_1 and λ_2 be two roots of the characteristic equation of the Jacobian matrix $J|_{(x,y)}$, then we have the following Definition and two Lemmas [4,10].

Definition 2.1. A fixed point (x, y) is called (i) sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is locally asymptotically stable; (ii) source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is locally unstable; (iii) saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or ($|\lambda_1| < 1$ and $|\lambda_2| > 1$); (iv) non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma 2.1. *Let $F(\lambda) = \lambda^2 + P\lambda + Q$, suppose that $F(1) > 0$, λ_1 and λ_2 are two roots of $F(\lambda) = 0$, then*

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $Q < 1$;
(ii) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$(*or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$) *if and only if* $F(-1) < 0$;
(iii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $Q > 1$;
(iv) $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $P \neq 0, 2$;
(v) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = 1$, $|\lambda_2| = 1$ *if and only if* $P^2 - 4Q < 0$ *and* $Q = 1$.

Lemma 2.2. *Assume that A is a* 2×2 *matrix, then the following results hold.*

(i) *All eigenvalues* λ *of A satisfy that* $|\lambda| < 1$ *if and only if* $|trA| - 1 < detA < 1$.
(ii) *Assume that* $|trA| - 1 = detA$,
 (a) *if* $trA > 0$, *then the eigenvalues of A are* $\lambda = 1$ *and* $\lambda = detA$;
 (b) *if* $trA < 0$, *then the eigenvalues of A are* $\lambda = -1$ *and* $\lambda = -detA$.
(iii) *Assume that* $|trA| - 1 \leqslant detA = 1$, *then the eigenvalues of A are* $\lambda = e^{\pm i\omega}$, *where* $\omega = \cos^{-1}(\frac{trA}{2})$.

The following partial results on fixed points and Jacobian matrix of the system (1.1) can also be found in [1], which is repeated here for convenience. The system (1.1) has three fixed points as $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = \left(\frac{r}{a+r}, \frac{r}{a+r}\right)$. These three fixed points are called extinction (exclusion, coexistence) fixed points respectively. P_3 is the unique positive fixed point (coexistence fixed point). The Jacobian matrix of the system (1.1) evaluated at (x, y) is as follows:

$$J|_{(x,y)} = \begin{bmatrix} 1 + r - 2rx - ay & -ax \\ ay & 1 + ax - 2ay \end{bmatrix},$$