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Numerical solution for fractional variational problems using the Jacobi polynomials

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A B S T R A C T

We exhibit a numerical method to solve fractional variational problems, applying a decomposition formula based on Jacobi polynomials. Formulas for the fractional derivative and fractional integral of the Jacobi polynomials are proven. By some examples, we show the convergence of such procedure, comparing the exact solution with numerical approximations.

1. Fractional variational calculus

Variational calculus deals with optimization problems for functionals depending on some variable function y and some derivative of y (see e.g. [1,2]). In many cases, the dynamic of such trajectories are not described by integer-order derivatives, but by real-order derivatives [3,4]. Solving these kind of problems usually implies finding the solutions of a fractional differential equation, the so-called Euler–Lagrange equation [5–10]. The main problem that arises with this approach is that in most cases there is no way to determine the exact solution. To overcome this situation, many numerical methods are being developed at this moment for fractional problems. One of the more commonly used methods consists in approximating the function by a polynomial y_n and the fractional derivative of y by the fractional derivative of y_n, and by doing this we rewrite the initial problem in a way such that applying already known methods from numerical analysis we can determine the optimal solution.

The variational problem that we address in this paper is stated in the following way. Given $\alpha, \beta \in (0,1)$, determine the minimizers of

$$J[y] = \int_a^b F(x, y(x), {}_aD_x^\alpha y(x), {}_aI_x^\beta y(x))dx, \tag{1}$$

under the constraint

$${}_aI_x^{1-\alpha} y(x)|_{x=b} = y_b. \tag{2}$$

Here, ${}_aD_x^\alpha y(x)$ denotes the Riemann–Liouville fractional derivative of y of order α,

$${}_aD_x^\alpha y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} y(t)dt,$$

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and ${}_aI_x^\beta y(x)$ the Riemann–Liouville fractional integral of y of order β,

$$ {}_aI_x^\beta y(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} y(t)dt. $$

We note that since function y is continuous, the condition ${}_aI_x^{1-\alpha} y(x)|_{x=a} = 0$ appears implicitly.

2. Numerical method

In this section we present a numerical method to solve the problem presented in Eqs. (1) and (2). We can find several methods in the literature to solve fractional problem types [11–13]. Our main idea is described in the following way: by using the Jacobi polynomials, the initial problem is converted into a non-linear programming problem, without dependence of fractional derivatives and fractional integrals. By doing this, we are able to find an approximation for the minimizers of the functional. To start, we briefly review some basic definitions of Jacobi polynomials.

2.1. Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ of indices $\alpha,\ \beta$ and degree n are defined by

$$ P_n^{(\alpha,\beta)}(t) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(1+\beta)_n(1+\alpha+\beta)_{n+k}}{k!(n-k)!(1+\beta)_k(1+\beta+\alpha)_n} \left(\frac{t+1}{2}\right)^k, \tag{3} $$

where $\alpha, \beta > -1$ are real parameters and

$$ (a)_0 = 1, \quad (a)_i = a(a+1)\dots(a+i-1), $$

The Jacobi polynomials are mutually orthogonal over the interval $(-1,1)$ with respect to the weight function $w^{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$. The Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ reduce to the Legendre polynomials $P_n(t)$ for $\alpha = \beta = 0$, and to the Chebyshev polynomials $T_n(t)$ and $U_n(t)$ for $\alpha = \beta = \mp 1/2$, respectively [14].

Another useful definition of the Jacobi polynomials of indices $\alpha,\ \beta$ and degree n is as [14,15]:

$$ P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^n}{dt^n}\left[(1-t)^{\alpha+n}(1+t)^{\beta+n}\right]. \tag{4} $$

This is a direct generalization of the Rodrigues formula for the Legendre polynomials, to which it reduces for $\alpha = \beta = 0$.

To present our numerical method, we use three interesting theorems as follows.

Theorem 2.1 [16]. *Let $\alpha > 0$ be a real number and $x \in [a,b]$. Then,*

$$ {}_aD_x^\alpha\left[(x-a)^\alpha P_k^{(0,\alpha)}\left(\frac{2(x-a)}{b-a}-1\right)\right] = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} P_k^{(\alpha,0)}\left(\frac{2(x-a)}{b-a}-1\right). $$

Theorem 2.2. *Let $\alpha - \beta > -1,\ \beta > -1$ be two real numbers and $x \in [a,b]$. Then,*

$$ {}_aD_x^\beta\left[(x-a)^\alpha P_k^{(0,\alpha)}\left(\frac{2(x-a)}{b-a}-1\right)\right] = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\beta+1)}(x-a)^{\alpha-\beta} P_k^{(\beta,\alpha-\beta)}\left(\frac{2(x-a)}{b-a}-1\right). $$

Proof 2.1. By substituting $t = \frac{2(x-a)}{b-a} - 1$ in (3), we get

$$ \xi(x) := (x-a)^\alpha P_k^{(0,\alpha)}\left(\frac{2(x-a)}{b-a}-1\right) = \sum_{m=0}^{k} \frac{(-1)^{k-m}(1+\alpha)_{k+m}}{m!(k-m)!(1+\alpha)_m}\frac{(x-a)^{m+\alpha}}{(b-a)^m} \tag{5} $$

Taking the Riemann–Liouville fractional derivative of order α on both side of (5), we conclude

$$ \begin{aligned} {}_aD_x^\beta \xi(x) &= \sum_{m=0}^{k} \frac{(-1)^{k-m}(1+\alpha)_{k+m}\Gamma(m+\alpha+1)}{m!(k-m)!(1+\alpha)_m\Gamma(m+\alpha-\beta+1)}\frac{(x-a)^{m+\alpha-\beta}}{(b-a)^m} \\ &= \frac{(1+\alpha)_k\Gamma(\alpha+1)(x-a)^{\alpha-\beta}}{(1+\alpha-\beta)_k\Gamma(\alpha-\beta+1)} \sum_{m=0}^{k} \frac{(-1)^{k-m}(1+\alpha-\beta)_k(1+\alpha)_{k+m}}{m!(k-m)!(1+\alpha-\beta)_m(1+\alpha)_k}\left(\frac{x-a}{b-a}\right)^m \\ &= \frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha-\beta+1)}(x-a)^{\alpha-\beta} P_k^{(\beta,\alpha-\beta)}\left(\frac{2(x-a)}{b-a}-1\right), \end{aligned} $$

and the proof is completed. □