Applied Mathematical Modelling 39 (2015) 6543–6552

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Nonlocal fractional order differential equations with changing-sign singular perturbation

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ARTICLE INFO

Article history:
Received 4 June 2013
Received in revised form 5 January 2015
Accepted 2 February 2015
Available online 24 February 2015

Keywords:
Nonlocal fractional order differential equations
Positive solutions
Sign-changing perturbation
Schauder's fixed point theorem

ABSTRACT

In this paper, we study the existence of positive solutions for a class of nonlocal fractional order differential equations with changing-sign singular perturbation. By means of Schauder's fixed point theorem, the conditions for the existence of positive solutions are established respectively for the cases where the nonlinearity is positive, negative and semipositone.

1. Introduction

Differential equation models can describe many nonlinear phenomena in applied mathematics, economics, finance, engineering, physical and biological processes [1–5]. In recent years, fractional order differential equations arise in the modeling of many complex dynamic phenomena in viscoelasticity, rheology, fluid mechanics, electrical networks and chemical physics, and thus have attracted more and more attention in the scientific research communities [6–18].

Recently, Li et al. [6] studied the nonlinear differential equation of fractional order

$$\mathscr{D}_t^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\quad 0<\alpha\leqslant 2,$$

subject to the boundary conditions

$$u(0)=0,\quad \mathscr{D}_t^{\beta}u(1)=a\mathscr{D}_t^{\beta}u(\xi).$$

By using some fixed point theorems, the existence and multiplicity of positive solutions to the problem were established for the case where $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous. In [7], Rehman and Khan investigated the following multi-point boundary value problem for fractional differential equation where the nonlinear terms and boundary conditions involve fractional derivatives of the unknown function

$$\begin{cases} \mathscr{D}_t^{\alpha}y(t)=f(t,y(t),\mathscr{D}_t^{\beta}y(t)),\quad t\in(0,1),\\ y(0)=0,\quad \mathscr{D}_t^{\beta}y(1)-\sum_{i=1}^{m-2}\zeta_i\mathscr{D}_t^{\beta}y(\xi_i)=y_0, \end{cases}\tag{1.1}$$

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http://dx.doi.org/10.1016/j.apm.2015.02.005
0307-904X/

where $1 < \alpha \leqslant 2, 0 < \beta < 1, 0 < \xi_i < 1, \zeta_i \in [0, +\infty)$ with $\sum_{i=1}^{m-2} \zeta_i \xi_i^{\alpha-\beta-1} < 1$. By using the Schauder fixed point theorem and the contraction mapping principle, the authors established the existence and uniqueness of nontrivial solutions for the BVP (1.1) provided that the nonlinear function $f : [0,1] \times \mathbb{R} \times \mathbb{R}$ is continuous and satisfies certain growth conditions. Recently, a coupled system of nonlinear fractional differential equations with multi-point boundary conditions was studied by Zhang et al. [8]

$$\begin{cases} -\mathscr{D}_t^{\alpha} x(t) = f(t, x(t), \mathscr{D}_t^{\beta} x(t), y(t)), \quad -\mathscr{D}_t^{\gamma} y(t) = g(t, x(t)), \quad t \in (0,1), \\ \mathscr{D}_t^{\beta} x(0) = 0, \quad \mathscr{D}_t^{\mu} x(1) = \sum_{j=1}^{p-2} a_j \mathscr{D}_t^{\mu} x(\xi_j), \quad y(0) = 0, \quad \mathscr{D}_t^{\nu} y(1) = \sum_{j=1}^{p-2} b_j \mathscr{D}_t^{\nu} y(\xi_j), \end{cases} \tag{1.2}$$

where $1<\gamma<\alpha\leqslant 2, 1<\alpha-\beta<\gamma, 0<\beta\leqslant\mu<1, 0<\nu<1, 0<\xi_1<\xi_2<\cdots<\xi_{p-2}<1, a_j, b_j\in[0,+\infty)$ with $\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\mu-1}<1$, $\sum_{j=1}^{p-2} b_j \xi_j^{\gamma-1}<1$. By using the fixed point theorem of the mixed monotone operator, the existence and uniqueness of a positive solution for the fractional differential Eq. (1.2) was established.

Motivated by the results mentioned above, in this paper, we study the existence of positive solutions for the following singular boundary value problem of fractional differential equation with a changing-sign term

$$\begin{cases} -\mathscr{D}_t^{\alpha+2} y(t) + \mathscr{D}_t^{\alpha} y(t) = f(t, y(t), \mathscr{D}_t^{\alpha} y(t)) + e(t), \quad 0 < t < 1, \\ a\mathscr{D}_t^{\alpha} y(0) - b\mathscr{D}_t^{\alpha+1} y(0) = \sum_{j=1}^{m-2} a_j \mathscr{D}_t^{\alpha} y(\xi_j), \\ c\mathscr{D}_t^{\alpha} y(1) + d\mathscr{D}_t^{\alpha+1} y(1) = \sum_{j=1}^{m-2} b_j \mathscr{D}_t^{\alpha} y(\xi_j), \end{cases} \tag{1.3}$$

where $0 < \alpha \leqslant 1, a, c \geq 0,\ b, d > 0, 0 < \xi_j < 1, a_j, b_j \in [0, +\infty), f : [0,1] \times [0,\infty) \times (0,\infty) \to (0,\infty)$ is continuous and may be singular near the zero for the third argument, $e \in L^1([0,1], R)$ may be sign-changing. So far, the effect of the perturbed term e on the existence of solutions to problem (1.3) is unknown. Hence the aim of this paper is to discuss this issue.

The rest of paper is organized as follows. In section two, we introduce some basic definitions and lemmas to be used for the development of our main results. In section three, our main results are summarized by three theorems, establishing the conditions for the existence of solutions to problem (1.3) under three different cases. In section four, an example is given to demonstrate the application of our results.

2. Basic definitions and preliminaries

Throughout this paper, we denote by $E = C[0,1]$ the Banach space of all continuous functions on [0,1] with the usual maximum norm $||x|| = \max_{0\leqslant t\leqslant 1} |x(t)|$. Let

$$C^+[0,1] = \{x \in E : x(t) \geqslant 0, \quad t \in [0,1]\},$$

then $C^+[0,1]$ is a normal cone in the Banach space E. Thus the space Ecan be equipped with a partial order given by

$$x, y \in E,\ x - y \in C^+[0,1] \Longleftrightarrow x(t) \geqslant y(t), \quad \text{for} \quad t \in [0,1].$$

In the following, we recall some lemmas of fractional calculus and then give some assumptions to be used later in this paper.

Definition 2.1 [2,5]. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 [2,5]. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathscr{D}_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(s) ds,$$

where $n = [\alpha] + 1$ in which $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.