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A posteriori global error estimator based on the error in the constitutive relation for reduced basis approximation of parametrized linear elastic problems

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ABSTRACT

In this paper we introduce a posteriori error estimator based on the concept of error in the constitutive relation to verify parametric models computed with a reduced basis approximation. We develop a global error estimator which leads to an upper bound for the exact error and takes into account all the error sources: the error due to the reduced basis approximation as well as the error due to the finite element approximation. We propose an error indicator to measure the quality of the reduced basis approximation and we deduce an error indicator on the finite element approximation.

1. Introduction

Finite Element Method is a common tool used to analyze and design parametrized mechanical systems. However, when a large set of parameters needs to be introduced in the model the computational effort increases drastically and many authors have recently shown interest in developing model reduction methods that exploit the fact that the response of complex models can often be approximated by the projection of the initial model on a low-dimensional reduced basis [1–4]. Reduced basis methods aim at speeding up the computational time for complex numerical models. They are based on an offline/online computational strategy which consist in determining in a first step a set of snapshots or a reduced basis (offline computations) that will be able to represent accurately the solutions for the problem studied. Different techniques are used to generate this basis, the more commonly found in the literature are the proper orthogonal decomposition and the greedy sampling approach [2,5]. In both case, the number of terms in the reduced basis is assumed to be very small compared to the number of degree of freedom of the finite element computation. Then, the approximate solutions of the parametrized problem are computed via performing a Galerkin projection onto the reduced basis space (online computations).

However, the accuracy of the obtained solutions depends on the quality of the mesh used as well as on the quality of the chosen reduced basis. If we denote by $\boldsymbol{\mu}$ the vector of parameters, the global error $\mathbf{e}_g$ is defined for any $\boldsymbol{\mu}$ by

$$\mathbf{e}_g(\boldsymbol{\mu}) = \mathbf{u}(\boldsymbol{\mu}) - \mathbf{u}_{rb}(\boldsymbol{\mu}), \tag{1}$$

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where $\mathbf{u}(\boldsymbol{\mu})$ is the exact solution of the parametrized problem and $\mathbf{u}_{rb}(\boldsymbol{\mu})$ is its reduced basis approximation. This global error can be split into two parts:

$$\mathbf{e}_g(\boldsymbol{\mu}) = \mathbf{u}(\boldsymbol{\mu}) - \mathbf{u}_h(\boldsymbol{\mu}) + \mathbf{u}_h(\boldsymbol{\mu}) - \mathbf{u}_{rb}(\boldsymbol{\mu}) = \mathbf{e}_h(\boldsymbol{\mu}) + \mathbf{e}_{rb}(\boldsymbol{\mu}) \tag{2}$$

where $\mathbf{u}_h(\boldsymbol{\mu})$ is the finite element solution of the parametrized problem and $\mathbf{e}_{rb}(\boldsymbol{\mu}) = \mathbf{u}_h(\boldsymbol{\mu}) - \mathbf{u}_{rb}(\boldsymbol{\mu})$ (resp. $\mathbf{e}_h(\boldsymbol{\mu}) = \mathbf{u}(\boldsymbol{\mu}) - \mathbf{u}_h(\boldsymbol{\mu})$) is the error due to the projection of the finite element solution in the reduced basis space (resp. the error due to the finite element approximation). Within the framework and reduced basis approximations based on proper orthogonal decomposition or greedy sampling methods, many works have been devoted to the computation of a posteriori error estimators to measure the error due to the projection of the finite element solution in the reduced basis space. An error estimator is proposed for elliptic partial differential equations in [1,6], for parabolic problems in [7,8], for computational homogenization in [9], for stochastic computations [10]. However, the proposed error estimators are focused on the estimation of the error due to the reduced basis approximation $\mathbf{e}_{rb}$and assume that the error due to the finite element approximation $\mathbf{e}_h$ is negligible. Within the framework of the proper generalized decomposition, a global error estimator based on the concept of error in the constitutive relation [11], has been recently proposed for transient thermal problems in [12], and for linear elastic problems in [13]. This error estimator requires to develop a double reduced basis approximation during the offline step, a kinematic approach and a static approach, for solving the parametrized problem.

In this paper, we focus on parametrized linear elastic models where the parametric bilinear form a is depending on parameters-dependent functions in an affine manner. The objective of this paper is to extend the constitutive relation error estimator to reduced basis approximations based on greedy sampling. The use of the constitutive relation error (CRE) requires the computation of an admissible pair $(\hat{\mathbf{u}}, \hat{\boldsymbol{\sigma}})$ for any parameter $\boldsymbol{\mu}$ during the online computations. Unlike the approach proposed in [12,13], we use directly the initial reduced basis (i.e. the reduced basis used to compute the solution $\mathbf{u}_{rb}$) to build the admissible pair $(\hat{\mathbf{u}}, \hat{\boldsymbol{\sigma}})$ and we do not need to compute a second reduced basis by a greedy sampling algorithm. Additionally, two error indicators are developed to separate in the global error estimator the part of the error due to the finite element approximation from the part due to the reduced basis approximation. This family of error estimators allows to construct error bounds of the energy norm and has been applied to separate the contribution of the different sources of error in finite element computations for non-linear problems [14–16], for domain decomposition problems [17,18]. We show that the global error estimator and the reduced basis error estimator are upper bounds of the corresponding exact errors. To compute efficiently the error estimates within the framework of an offline/online reduced basis method we need a further assumption, and we assume that the complementary energy is, as well as the bilinear form a, depending on parameters-dependent functions in an affine manner.

The paper is organized as follows: In Section 2, we introduce parametric problem to be solved, we briefly recall the reduced basis methodology, and we define the approximation errors introduced. The formulation of the global error estimator, the finite element error indicator and the reduced basis error indicator, as well as the offline/online strategy to compute them, are described in Section 3. Finally, in Section 4 the different errors are analyzed through a numerical example.

2. Problem to be solved

2.1. Linear elastic model

Let us consider an elastic structure defined in a domain Ω bounded by Γ. The external actions on the structure are represented by a surface force density $\mathbf{T}$ defined over a subset Γ_N of the boundary and a body force density $\mathbf{b}$ defined in Ω. We assume that a prescribed displacement $\mathbf{u} = \mathbf{u}_d$ is imposed on $\Gamma_D = \partial\Omega - \Gamma_N$. The material is assumed to be linear elastic, being $\mathbf{C}$the Hooke tensor. We consider that the problem is dependent of a set of parameters $\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p$. The problem can be formulated as: find a displacement field $\mathbf{u}(\boldsymbol{\mu}) \in \mathcal{U}$ and a stress field $\boldsymbol{\sigma}(\boldsymbol{\mu})$ defined in Ω which verify:

- the kinematic constraints:

$$\mathbf{u}(\boldsymbol{\mu}) = \mathbf{u}_d \text{ on } \Gamma_D \tag{3}$$

- the equilibrium equations:

$$\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{\mu}) + \mathbf{b}(\boldsymbol{\mu}) = \mathbf{0} \text{ in } \Omega \text{ and } \boldsymbol{\sigma}(\boldsymbol{\mu})\mathbf{n} = \mathbf{T}(\boldsymbol{\mu}) \text{ in } \Gamma_N \tag{4}$$

- the constitutive equation:

$$\boldsymbol{\sigma}(\boldsymbol{\mu}) = \mathbf{C}(\boldsymbol{\mu})\boldsymbol{\varepsilon}(\mathbf{u}(\boldsymbol{\mu})) \text{ in } \Omega \tag{5}$$

$\mathbf{n}$ denotes the outgoing normal to Ω. $\mathcal{U}$ is the space in which the displacement field is being sought, $\mathcal{U}^0$ the space of the fields in $\mathcal{U}$ which are zero on Γ_D, and $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized deformation associated with the displacement: $[\boldsymbol{\varepsilon}(\mathbf{u})]_{ij} = 1/2\,(u_{i,j} + u_{j,i})$.

The strong form of the problem (3-5) is equivalent to the classical weak form formulation: find $\mathbf{u} \in \{\mathbf{v} \in \mathcal{U};\ \mathbf{v}_{|\Gamma_D} = \mathbf{u}_d\}$ such that:

$$a(\mathbf{u}(\boldsymbol{\mu}), \mathbf{u}^*; \boldsymbol{\mu}) = f(\mathbf{u}^*; \boldsymbol{\mu}) \quad \forall \mathbf{u}^* \in \mathcal{U}^0 \tag{6}$$