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Multiple sparse-grid Gauss–Hermite filtering

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ABSTRACT

A new method for nonlinear estimation, based on sparse-grid Gauss–Hermite filter (SGHF) and state-space partitioning, termed as Multiple sparse-grid Gauss–Hermite filter (MSGHF) is proposed in this work. Gauss–Hermite filter is a widely acclaimed filtering technique for its high accuracy. But the computational load associated with it is so high, that it becomes difficult to apply it on-board for higher dimensional problems. SGHF showcased comparable performance with the GHF, with less computational burden. The proposed technique, MSGHF, further reduces the computational burden considerably, with the filter accuracy remaining almost the same. Simulation results illustrate the performance of the proposed filter with respect to GHF and SGHF.

1. Introduction

Nonlinear filtering problems are frequently encountered in various real life scenarios like navigation [1], target tracking [2], weather forecasting [3], economics [4] etc. Filtering involves the recursive estimation of unknown states of a system by making use of noisy measurements. All these filtering problems are generally addressed under the Bayesian estimation framework. The discrete state-space model of a dynamic system, which includes both process and measurement model can be described as

$$x_k = \phi(x_{k-1}) + \eta_{k-1} \tag{1}$$

and

$$y_k = \gamma(x_k) + v_k \tag{2}$$

respectively. Here $x_k \in \Re^n$ represents the unknown states of the system, $y_k \in \Re^p$ denotes the measurement at any instant k, where $k = 0, 1, 2, \ldots, N$. $\phi(x_k)$ and $\gamma(x_k)$ are given nonlinear functions of x_k and k. The process and measurement noises are given by $\eta_k \in \Re^n$ and $v_k \in \Re^p$ respectively. These are assumed to be uncorrelated and normally distributed with zero mean and covariance Q_k and R_k, respectively.

Under the Bayesian framework, the two steps involved in filtering process are:

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(1) Prediction step: In this step, the prior probability density function, $p(x_k|y_{1:k-1})$ is obtained using the Chapman–Kolmogorov equation,

$$p(x_k|y_{1:k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|y_{1:k-1})dx_{k-1}. \tag{3}$$

(2) Update step: In this step, the new measurements y_k and the prior density function is used to obtain the posterior probability density function using Bayes' rule,

$$p(x_k|y_{1:k}) = \frac{p(y_k|x_k)p(x_k|y_{1:k-1})}{\int p(y_k|x_k)p(x_k|y_{1:k-1})dx_k}. \tag{4}$$

There is no method for obtaining the solution of above Eqs. (3) and (4) when the process and measurement is nonlinear and the pdf encountered is non-Gaussian. This is because of the fact that the integrals encountered becomes intractable [5–8]. One approach is to approximate with Gaussian distribution and take the mean as a point estimate. Under this assumption, the prior and posterior probability densities can be expressed as

$$p(x_k|y_{1:k-1}) = \aleph(\hat{x}_{k|k-1}, P_{k|k-1})$$

and

$$p(x_k|y_{1:k}) = \aleph(\hat{x}_{k|k}, P_{k|k}),$$

where $\aleph(\hat{x}_{k|k}, P_{k|k})$ represents the normal distribution with mean $\hat{x}_{k|k}$ and covariance $P_{k|k}$. To obtain prior and posterior probability densities, Eqs. (3) and (4) need to be solved.

Literature about early suboptimal algorithms for nonlinear filtering begins with the extended Kalman filter (EKF) [6]. However, it resulted in undesirable performances like poor tracking accuracy or divergence in estimation error [7–9]. Later, various nonlinear filtering algorithms such as the unscented Kalman filter (UKF) [9–12], cubature Kalman filter (CKF) [13] and it's variants [14,15], cubature quadrature Kalman filter (CQKF) [16] and its variants [17] *etc.* were introduced and these filters performed with acceptable accuracy. To achieve more accuracy, Gauss–Hermite filter (GHF) was also introduced [18,19]. It makes use of the Gauss–Hermite quadrature rule and has the highest accuracy among all the above mentioned filters. But, it suffers from the *curse of dimensionality* problem since the number of quadrature points required increases exponentially with the increase in dimension of the system. So it is difficult to apply it on-board for higher dimensional problems.

Sparse-grid quadrature filter (SGQF)[20] is an efficient filtering algorithm which can achieve accuracy levels almost as high as GHF, but with very less computational load. It uses the Smolyak rule [21] for extending the one-dimensional quadrature rule to multi-dimensional problems. This formulation considerably decreases the computational load in the algorithm. Further reduction in computational burden was achieved by multiple quadrature Kalman filter (MQKF)[22]. It consists of state-space partitioning technique and runs several filters in parallel.

In this paper, we combine the idea of MQKF and SGHF which results in an efficient nonlinear filtering technique in terms of computational cost. SGHF itself is efficient in reducing the computational cost since the number of points required increases only as a function of polynomial of dimension of the system. When applied with state-space partitioning technique, it further reduces the computational cost, without hampering the accuracy measures considerably.

2. Sparse-grid Gauss–Hermite filter

Sparse-grid Gauss–Hermite filter is computationally more efficient than GHF. It uses Gauss–Hermite quadrature rule for generating univariate quadrature points and its multi-dimensional extension is obtained using Smolyak rule [21]. In short, filter uses weighted sparse-grid quadrature points to evaluate the multi-dimensional intractable integrals encountered in the nonlinear Bayesian estimation problem. GHF uses product rule for converting univariate points to its multi-variate extension, while SGHF uses a linear combination of tensor products to do the same. This gives an additional advantage for SGHF that the accuracy level of estimation can be defined separately.

2.1. Univariate quadrature point generation

In this work, the univariate quadrature points are generated using the technique used in [18], which was initially proposed by Golub [23]. The unknown probability densities are considered as Gaussian and approximation is done by defining a set of Gauss–Hermite quadrature points and their corresponding weights.

Consider an integral of any function $f(x)$,

$$I = \int_{-\infty}^{\infty} f(x)e^{-x^2}dx.$$

It can be evaluated numerically with N quadrature points as

$$I \approx \sum_{j=1}^{N} f(q_j)w_j,$$