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The numerical solution of advection–diffusion problems using new cubic trigonometric B-splines approach

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ABSTRACT

A new cubic trigonometric B-spline collocation approach is developed for the numerical solution of the advection–diffusion equation with Dirichlet and Neumann's type boundary conditions. The approach is based on the usual finite difference scheme to discretize the time derivative while a cubic trigonometric B-spline is utilized as an interpolation function in the space dimension with the help of θ-weighted scheme. The present scheme stabilizes the oscillations that are normally displayed by the approximate solution of the transient advective–diffusive equation in the locality of sharp gradients produced by transient loads and boundary layers. The scheme is shown to be stable and the accuracy of the scheme is tested by application to various test problems. The proposed approach is numerically verified to second order and shown to work for the Péclet number ≤ 5.

1. Introduction

The advection–diffusion equation (sometime called the convection–diffusion equation) plays an important role in various physical systems involving fluid flow [1]. It can describe physical phenomena where energy is transformed during two processes advection and diffusion [2]. It is a one dimensional parabolic partial differential equation which illustrates advection and diffusion of quantities such as mass, energy, heat, vorticity etc. [3]. It can be used to depict heat transfer in a draining film [4], volumetric concentration of a pollutant [5], dispersion of tracers in porous media [6], water transfer in soils [7], the spread of pollutants in rivers and streams [8] as well as long range transport of pollutants in the atmosphere [9]. The mathematical formulation of an advection–diffusion problem is:

$$\frac{\partial C(x,t)}{\partial t} + \beta \frac{\partial C(x,t)}{\partial x} = \alpha \frac{\partial^2 C(x,t)}{\partial x^2}, \qquad 0 \leq x \leq L, \quad 0 < t \leq T \tag{1}$$

with initial conditions:

$$C(x, t = 0) = C_0(x), \tag{2}$$

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and following two types of boundary conditions:

1. **Dirichlet boundary conditions:**

$$\begin{aligned} C(x=0,\ t) &= f_0(t), \\ C(x=L,\ t) &= f_L(t). \end{aligned} \tag{3}$$

2. **Neumann's boundary conditions:**

$$\begin{aligned} C_x(x=0,\ t) &= g_0(t), \\ C_x(x=L,\ t) &= g_L(t), \end{aligned} \tag{4}$$

where the parameter β shows the advection velocity and α is the diffusion coefficient. Here, $C = C(x, T)$ is concentration of a quantity at a point x at time T and $C_0(x), f_0(t), f_L(t), g_0(t), g_L(t)$ are known functions.

Several numerical methods have developed for the solution of one dimensional advection–diffusion problems and many are based on two level finite difference approximations [10–20]. Dehghan [3] has developed several numerical approaches based on fully explicit and implicit finite difference approximations for the solution of three dimensional advection–diffusion problems. High order compact boundary value method based on method of lines has been discussed in [21] for solving unsteady advection–diffusion problems. Dehghan [22] has solved the one dimensional advection–diffusion problem using second order accurate practical designing approach whose application is based on the modified equivalent partial differential equation. Explicit and implicit schemes have been developed in [23] which were based on weighted finite difference approximations for the solution of advection–diffusion problem. A compact finite difference approximation of fourth-order was used in [24] to discretize spatial derivatives for solving the one-dimensional heat and advection–diffusion equations. The cubic C^1 spline collocation method was used for the resulting linear system of equations. Finite element and difference approximation methods were used in several numerical methods [25–33] for the discretization of time derivatives. In these methods different splines such as redefine cubic B-spline [2], exponential B-spline [25], cubic B-spline [5,30], spline functions [26], cubic B-spline differential quadrature methods [27], cubic spline interpolation [28], B-spline finite element [29,31] and Taylor–Galerkin quadratic and cubic B-spline finite element [32,33] were used as interpolating function in space dimension for solving one-dimensional advection–diffusion equations. For further details on other numerical methods for the solution of a one-dimensional advection–diffusion problem subject to different types of boundary conditions in the literature, see references [34–41].

In this paper, a new two-time level implicit approach is developed to approximate the numerical solution of the advection–diffusion problem (1) subject to initial constraints in Eq. (2) and Dirichlet and Neumann's type boundary conditions in Eqs. (3) and (4), respectively. The approach is based on new cubic trigonometric B-spline functions. A finite difference approach and θ-weighted scheme is used respectively for the time and space discretizations. The weighting parameter θ is used to determine both the approximation accuracy of the scheme and numerical stability. Some authors have used the exponential and polynomial B-spline collocation method for solving the advection–diffusion problem but not so far as we are aware, with the used of cubic trigonometric B-spline collocation method. Cubic trigonometric B-spline is used in this study to approximate the solution in the space direction. Our proposed approach is found to work for the Péclet number ≤ 5. It is shown that the present method is stable and the accuracy performance is investigated by application to various test problems. The obtained solutions are found to be in reasonable agreement with known exact solutions and further, more accurate than the methods developed by Dehghan [1], Mittal and Jain [2], Goh et al. [5], Douglas method, Mohebbi and Dehghan [24], Mohammad [25], Kadalbajoo and Arora [33], Chawla et al. [39] and Rizwan-Uddin [40]. The order of convergence is shown to be approximately equal to two.

This paper is structured as follows: a cubic trigonometric B-spline collocation approach is presented in Section 2. Approximate solution of the advection–diffusion problem is discussed in Section 3. The von Neumann approach is used to investigate the stability of method in Section 4. Several test problems are considered in Section 5 to show the feasibility of the proposed method. Finally, in Section 6, the conclusion of this study is given.

2. Cubic trigonometric B-spline collocation approach

In this approach, the space derivatives are approximated by using cubic trigonometric B-spline method (CuTBSM). A mesh Ω which is equally divided by knots x_i into N subintervals $[x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, N-1$ such that, $\Omega : a = x_0 < x_1 < \cdots < x_N = b$ is used. For the advection–diffusion equation, an approximate solution using collocation method with cubic trigonometric B-spline is obtained in the form [46]:

$$c(x,t) = \sum_{i=-3}^{N-1} \delta_i(t)\, TB_i(x), \tag{5}$$

where $\delta_i(t)$ are to be calculated for the approximated solutions $c(x,\ t)$ to the exact solutions $\bar{u}(x,t)$, at the point $(x_i,\ t_j)$. A C^2 piecewise cubic trigonometric B-spline basis functions $TB_i(x)$ over the uniform mesh can be defined as [43,44]:

$$TB_i(x) = \frac{1}{\omega}\begin{cases} p^3(x_i), & x \in [x_i, x_{i+1}], \\ p(x_i)\big(p(x_i)q(x_{i+2}) + q(x_{i+3})p(x_{i+1})\big) + q(x_{i+4})p^2(x_{i+1}), & x \in [x_{i+1}, x_{i+2}], \\ q(x_{i+4})\big(p(x_{i+1})q(x_{i+3}) + q(x_{i+4})p(x_{i+2})\big) + p(x_i)q^2(x_{i+3}), & x \in [x_{i+2}, x_{i+3}], \\ q^3(x_{i+4}), & x \in [x_{i+3}, x_{i+4}], \end{cases} \tag{6}$$