



# Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions



Mansur I. Ismailov\*, Muhammed Çiçek

Department of Mathematics, Gebze Technical University, Gebze 41400, Kocaeli, Turkey

## ARTICLE INFO

### Article history:

Received 16 May 2014

Revised 26 November 2015

Accepted 7 December 2015

Available online 18 December 2015

### Keywords:

Inverse source problem

Fractional diffusion equation

Generalized Fourier method

## ABSTRACT

In this paper, an inverse problem of determining a time-dependent source term in a one-dimensional time-fractional diffusion equation from the energy measurement is studied. This problem is obtained from a classical diffusion problem by replacing the time derivative with a fractional derivative. The well-posedness of the inverse problem is shown by using eigenfunction expansion of a non-self adjoint spectral problem along the generalized Fourier method.

© 2015 Elsevier Inc. All rights reserved.

## 1. Introduction

It is known that [1,2] the anomalous diffusion (subdiffusion, superdiffusion, non-Gaussian diffusion) phenomena show many different aspects from classical diffusion processes. The time-fractional diffusion equations are commonly used to describe for such an anomalous diffusion process. These type of equations are obtained by replacing the standard time derivative with a time fractional derivative and can be derived from continuous-time random walk [3] similar to the derivation of the classical diffusion equation from Brownian motion assumption.

The direct problems for fractional diffusion equations such as an initial or boundary value problems have been studied extensively in [4–7] and references therein. In contrast of direct problem, the mathematical analysis of inverse problem for the fractional diffusion equation is not satisfactorily investigated. The first mathematical results for the inverse problem of finding diffusion coefficient for a fractional differential equation are obtained in [8]. In this paper the uniqueness theorem is proved by using expansion in terms of eigenfunctions of suitable Sturm–Liouville problem along the Gelfand–Levitan theory. Similar eigenfunction expansion result along the analytic continuation and Laplace transform is used in determination of space-dependent source term in a fractional diffusion equation in [9]. Spectral analysis of suitable Sturm–Liouville operator is actively used in other coefficient identification problems for time-fractional diffusion problems, [10,11]. The papers [12] and [13] study inverse problems of finding space dependent and time-dependent source terms, respectively, in time-fractional diffusion equation by using eigenfunction expansion of the non-self adjoint spectral problem along the generalized Fourier method. Some of numerical aspects of inverse source problem in time fractional heat equation are studied in [14–17].

Consider the time-dependent heat conduction equation

$$D_{0+}^q (u(x, t) - u(x, 0)) = u_{xx} + r(t)f(x, t), \quad (x, t) \in \Omega_T, \quad (1)$$

\* Corresponding author. Tel.: +90 2626051641.

E-mail address: [mismailov@gtu.edu.tr](mailto:mismailov@gtu.edu.tr), [mcicek@gtu.edu.tr](mailto:mcicek@gtu.edu.tr), [mismailov@gyte.edu.tr](mailto:mismailov@gyte.edu.tr), [mansur.ismailov@gmail.com](mailto:mansur.ismailov@gmail.com) (M.I. Ismailov).

with the initial condition

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1, \quad (2)$$

the boundary condition

$$u(0, t) = u(1, t), \quad u_x(0, t) + \alpha u(0, t) = 0, \quad 0 \leq t \leq T, \quad (3)$$

where  $D_{0+}^q$  refers to the Riemann–Liouville fractional derivative of order  $q$  ( $0 < q < 1$ ) in the time variable defined by

$$D_{0+}^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{u(\tau)}{(t-\tau)^q} d\tau,$$

$\alpha$  is real constant,  $\varphi(x)$  is given initial temperature and  $f(x, t)$  is given source term,  $\Omega_T = \{(x, t) : 0 < x < 1, 0 < t \leq T\}$ . The choice of the term  $D_{0+}^q (u(x, \cdot) - u(x, 0))(t)$  instead of the usual term  $D_{0+}^q (u(x, \cdot))(t)$  is not only to avoid the singularity at zero, but also impose a meaningful initial condition (without fractional integral) [18].

If the function  $r(t)$  is known, the problem of finding  $u(x, t)$  from (1)–(3) is called the direct problem. However, the problem here is that the source function  $r(t)$  is unknown, which needs to be determined by energy condition

$$\int_0^1 u(x, t) dx = E(t), \quad 0 \leq t \leq T, \quad (4)$$

where  $E(t)$  are given functions. This problem is called the inverse problem.

A precedent model for a similar setup in microwave heating process used in various applications in industry, e.g. in ceramics and in food processing. The external energy is supplied to a target at a controlled level by the microwave generating equipment. However, the dielectric constant of the target material varies in space and time, resulting in spatially heterogeneous conversion of electromagnetic energy to heat. This can correspond to source term  $r(t)f(x, t)$  in (1), where  $r(t)$  is proportional to power of external energy source and  $f(x, t)$  is local conversion rate of microwave energy. It is needed to notice that this spatial variation of absorbing material does not greatly affect the thermal diffusivity, which is due to another material at higher concentration. It is also needed to say that the temperature is not so high that temperature dependence of dielectric constant is important, as in thermal runaway studies [19]. If  $u(x, t)$  denotes the concentration of absorbed energy in this example, then its integral over all volume of material determining the time dependence absorbed energy. The above mentioned inverse problem in the present paper for such a model gives an idea of how total energy content might be externally controlled.

The integral condition (4) arises naturally and can be used as supplementary information in the determination of the source term. Such type of condition can model various physical phenomena in context of chemical engineering [20,21], thermoelasticity [22], heat conduction and diffusion process [23–25], fluid flow in porous media [26]. In heat conduction process, the integral condition (4) is encountered in problems related to particle diffusion in turbulent plasma, and also in heat propagation in a thin rod in which the law of variation  $E(t)$  of the total quantity of heat in the rod is given [23]. A problem in the design of periodic contact in thermoelasticity formulated as an inverse problem with prescribed contact pattern, which is considered in [22], is reduced to a heat conduction problem with energy specification.

The paper [20] considers the model on certain chemicals absorb light at various frequencies. Consequently, the intensity of such light on a photoelectric cell gives an electric signal which is proportional to the total amount of chemical present in the volume through which the light passes. If  $u(x, t)$  denotes the concentration of such a chemical which is diffusing in a straight tube with  $x$  measured in the direction of the axis of the tube, then the electric signal produced by a light beam passing through the tube at right angles between  $x = 0$  and  $x = b$  is proportional to  $\int_0^b u(x, t) dx$ ,  $0 < b < 1$  which is the total mass of the chemical in  $0 \leq x \leq b$  at time  $t$ . If the boundaries of chemical are supported by the conditions (3) this model gives an idea how the limit  $\lim_{b \rightarrow 1^-} \int_0^b u(x, t) dx$  of total mass might be externally controlled.

On the other hand, use of integral condition (4) arises when the data on the boundary cannot be measured directly, but only the average value of the solution can be measured along the boundary. More precisely classical boundary conditions (Neumann, Dirichlet and Robin type) are not always adequate as it depends on the physical context which data can be measure at the boundary of the physical domain. The classical boundary conditions cases, one can have a selection of such large noise local space measurement, but which on average produce a less noisy non-local measurement (4). In the case when the equation is valid in boundary  $x = 0$  and  $x = 1$ , the nonlocal condition (4) is reduced to the local condition related with  $u_x(1, t) - u_x(0, t)$ . Taking the derivative  $D_{0+}^q$  in the condition (4) and in view of Eq. (1) in  $x = 0$  and  $x = 1$  we have

$$\int_0^1 D_{0+}^q (u(x, t) - u(x, 0)) dx = D_{0+}^q [E(t) - E(0)]$$

or

$$u_x(1, t) - u_x(0, t) = D_{0+}^q [E(t) - E(0)] - r(t) \int_0^1 f(x, t) dx.$$

The boundary conditions related with the difference  $u_x(1, t) - u_x(0, t)$  appears in some chemical and biological processes [27,28].

Download English Version:

<https://daneshyari.com/en/article/1703129>

Download Persian Version:

<https://daneshyari.com/article/1703129>

[Daneshyari.com](https://daneshyari.com)