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Sine-Cosine Wavelets Approach in Numerical Evaluation of Hankel Transform for Seismology

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ABSTRACT

The computation of electromagnetic (EM) fields for 1-Dlayered earth model requires evaluation of Hankel transform. In this paper we propose a stable algorithm for the first time that is quite accurate and fast for numerical evaluation of the Hankel transform using Sine-Cosine wavelets arising in seismology. We have projected an approach depending on separating the integrand $rf(r)J_\nu(pr)$ into two components; the slowly varying components $rf(r)$ and the rapidly oscillating component $J_\nu(pr)$. Then either $rf(r)$ is expanded into wavelet series using Sine-Cosine wavelets orthonormal basis and truncating the series at an optimal level or approximating $rf(r)$ by a quadratic over the subinterval using the Filon quadrature philosophy. The solutions obtained by proposed Sine-Cosine wavelet method applied on 5 test functions indicate that the approach is easy to implement and computationally very attractive. We have supported a new efficient and stable technique based on compactly supported orthonormal wavelet bases.

1. Introduction

Electromagnetic (EM) depth sounding is, under favourable conditions, extremely useful in petroleum exploration, groundwater exploration, permafrost thickness determination exploration of geothermal resources, and foundation engineering problems. However, for data interpretation one needs fast and efficient computations of geoelectromagnetic anomaly equations. These equations appear as Hankel Transform (HT) (also known as Bessel Transform) integral of the form [1]:

$$F_\nu(p) = \int_0^\infty r f(r) J_\nu(pr)\,dr, \tag{1}$$

and HT being self-reciprocal, its inverse is given by:

$$f(r) = \int_0^\infty p F_\nu(p) J_\nu(pr)\,dp, \tag{2}$$

where J_ν is the ν th-order Bessel function of first kind. This form of Hankel transform has the advantage of reducing to the Fourier Sine or Cosine transform when $\nu = \pm\frac{1}{2}$. Due to the oscillatory behaviour of $J_\nu(pr)$, standard quadrature methods applied to these integrals can be slow to convergence or may fail if the integral is divergent. Numerical evaluation of Hankel transforms is ubiquitous in the mathematical treatment of physical problems involving cylindrical symmetry, optics, electromagnetism and seismology. Many different types of algorithms and software have been developed to evaluate numerically

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Hankel transform integrals in Geophysics [2,3]. The ubiquity of these integrals in EM geophysics motivates the need for accurate and efficient numerical integration techniques.

From variety of algorithm, a potential user would probably find it difficult to select any one algorithm that might be best for a particular application [4]. For an overview of these algorithms and their numerical complexity, the reader is referred to [5,6]. From numerical analysis point of view, it is generally not possible to develop a single algorithm to solve every type of Hankel transform problem that arises.

Analytical evaluations are rare and hence numerical methods become important. The usual classical methods like Trapezoidal rule, cotes rule and so forth, connected with replacing the integrand by sequence of polynomials have high accuracy if integrand is smooth. But $rf(r)J_\nu(pr)$ and $pF_\nu(p)J_\nu(pr)$ are rapidly oscillating functions for large r and p, respectively. To overcome these difficulties, various different techniques are available in the literature. Several papers have been written to the numerical evaluation of the HT in general and the zeroth-order in particular [7–21]. There are two general methods of the effective calculation in this area. The first is the Fast Hankel transform [22]. The specification of that method is transforming the function to the logarithmical space and fast Fourier transform in that space. This method needs a smoothing of the function in log space. The second method is based on the separation of the integrand into product of slowly varying component and a rapidly oscillating Bessel function [23,24]. But it needs the smoothness of the slow component for its approximation by lower-order polynomials. There are several extrapolation methods developed in the eighties. In particular, the papers by Levin and Sidi [25,26] are relevant.

The organization of the paper is as follows:

Section 2 gives a brief description of Sine–Cosine wavelets followed by orthonormal basis of Sine-cosine wavelets in Section 3. Section 4 is devoted to function approximation by Sine-cosine basis. Section 5 gives numerical implementation of method followed by summary and conclusion in Section 6.

2. Properties of Sine–Cosine wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\varphi(t)$ called the mother wavelets. When the dilation parameter is 2 and the translation parameter is 1 we have the following family of discrete wavelets [27]:

$$\psi_{kn}(t) = 2^{\frac{k}{2}}\varphi(2^k t - n),$$

where ψ_{kn} form a wavelet orthonormal basis for $L^2(R)$.

Sine–Cosine wavelets $\psi_{n,m}(t) = \psi(n, k, m, t)$ have four arguments;

$n = 0, 1, 2, \ldots, 2^k - 1$, $k = 0, 1, 2, \ldots,$, the values of m are given in Eq. (4) and t is the normalized time. They are defined on the interval [0, 1) as:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} f_m(2^k t - n), & \frac{n}{2^k} \le t < \frac{n+1}{2^k} \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

with

$$f_m(t) = \begin{cases} \frac{1}{\sqrt{2}}, & m = 0 \\ \cos(2m\pi t), & m = 1, 2, \ldots, L, \\ \sin(2(m-L)\pi t), & m = L+1, L+2, \ldots, 2L, \end{cases} \tag{4}$$

It is clear that the set of Sine–Cosine wavelets also forms and orthonormal basis for $L^2([0, 1])$. An efficient and fast algorithm to compute Hankel transform of order $\nu > -1$ based on Sine–Cosine wavelets and numerical techniques is proposed in this paper.

3. Orthonormal Basis Functions

In this section orthonormal basis functions for sine-cosine wavelets are obtained by fixing $k = 1$ and $L = 2$:

$$\left.\begin{array}{l} \psi_{0,0}(t) = \sqrt{2} \\ \psi_{0,1}(t) = 2\cos(4\pi t) \\ \psi_{0,2}(t) = 2\cos(8\pi t) \\ \psi_{0,3}(t) = 2\sin(4\pi t) \\ \psi_{0,4}(t) = 2\sin(8\pi t) \end{array}\right\}, \quad 0 \le t < \frac{1}{2}$$

$$\left.\begin{array}{l} \psi_{1,0}(t) = \sqrt{2} \\ \psi_{1,1}(t) = 2\cos(2\pi(2t-1)) \\ \psi_{1,2}(t) = 2\cos(4\pi(2t-1)) \\ \psi_{1,3}(t) = 2\sin(2\pi(2t-1)) \\ \psi_{1,4}(t) = 2\sin(4\pi(2t-1)) \end{array}\right\}, \quad \frac{1}{2} \le t < 1 \tag{5}$$