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A high-order spectral method for the multi-term time-fractional diffusion equations

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ABSTRACT

The multi-term time-fractional diffusion equation is a useful tool in the modeling of complex systems. This paper aims to develop a high order numerical method for solving multi-term time-fractional diffusion equations. Based on the space-time spectral method, a high-order scheme is proposed in the present paper. In this method, the Legendre polynomials are adopted in temporal discretization and the Fourier-like basis functions are constructed for the spatial discretization. Such a space-time spectral method possesses high efficiency and exponential decay in both time and space directions. Rigorous proofs are given here for the stability and convergence of the scheme. Numerical results show good agreement with the theoretical analysis.

1. Introduction

The time-fractional diffusion or diffusion-wave equation is a useful tool for modeling a wide range of important physical phenomena. It can be written as:

$$ {}_0^C\!D_t^\alpha u(x,t) = K_\alpha \partial_{xx} u(x,t) + f(x,t), x \in (a,b), t \in (0,T], \tag{1.1} $$

where x and t are the space and time variables, K_α is a positive constant, $f(x, t)$ is a source term of sufficient smoothness, and ${}_0^C\!D_t^\alpha$ is the Caputo fractional derivative (see e.g. [1]). When $0 < \alpha < 1$, (1.1) is called the fractional diffusion equation, and when $0 < \alpha < 2$, (1.1) is called the fractional diffusion-wave equation. These equations can represent, for example, propagation of mechanical waves in viscoelastic media, transport in amorphous semiconductors, non-Markovian diffusion process with memory (see e.g. [2–5]).

It has been known that the behavior of some time-changed processes may be better described using variable-order differential operator rather than time-varying coefficients [6–8]. Lorenzo and Hartley developed the conception of variable- and distributed order fractional operators [9]. The mathematical analysis and physical meanings on these operators were then investigated by several authors (see e.g. [10–13]). Using the distributed-order derivative, (1.1) is converted into (see

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[9]):

$$D_t^{w(\alpha)}u(x,t) = K_\alpha \partial_{xx}u(x,t) + f(x,t), x \in (a,b), t \in (0,T],$$

where

$$D_t^{w(\alpha)}u(x,t) = \int_I {}_0^C\mathcal{D}_t^\alpha u(x,t)w(\alpha)d\alpha,$$

and $I = [0,1] or [0,2]$, corresponding to diffusion or diffusion-wave equation, respectively, with the weight function w satisfying:

$$w(\alpha) \geq 0, \int_I w(\alpha)d\alpha = W(\text{positive constant}).$$

An important particular case of the time-fractional diffusion or diffusion-wave equation of distributed order is the multi-term time-fractional diffusion or diffusion-wave equation, whose weight function is taken into the linear combination of the Dirac δ-functions, i.e.,

$$w(\alpha) = \sum_{i=1}^n d_i \delta(\alpha - \alpha_i).$$

The multi-term time-fractional diffusion and diffusion-wave equations may be written as (see e.g. [14,15] and references therein):

$$P_{\alpha_1,\ldots,\alpha_n,\alpha}(D_t)u(x,t) = K_\alpha \partial_{xx}u(x,t) + f(x,t), \tag{1.2}$$

where

$$P_{\alpha_1,\alpha_2,\ldots,\alpha_n,\alpha}(D_t)u(x,t) = \left({}_0^C\mathcal{D}_t^\alpha + \sum_{i=1}^n d_i\, {}_0^C\mathcal{D}_t^{\alpha_i}\right)u(x,t),$$

with $0 < \alpha_1 < \cdots < \alpha_n < \alpha < 2$ or $0 < \alpha_1 < \cdots < \alpha_n < \alpha < 1$, and K_α, d_i are some positive constants.

Eq. (1.2) is not only a useful tool for modeling the behavior of viscoelastic fluid and rheological material [9], but also often appears while discretizing the distributed-order derivative in approximating the distributed-order differential equations [16]. Hence, studies on the multi-term time-fractional differential equations have become important and useful for different applications.

Studies of analytical solution to the multi-term time-fractional equations have been carried out by some authors. Daftardar-Gejji and Bhalekar considered the multi-term time-fractional diffusion-wave equation using the method of separation of variables [17]. Luchko [15] studied the well-posedness of the multi-term time-fractional diffusion equation based on an approximate maximum principle. The solution of the diffusion-wave problem in one dimension with orders belonging to the intervals (0, 1), (1, 2) and (0, 2), respectively, was found in [18]. Jiang et al. studied the multi-term time-space fractional advection-diffusion equation based on the spectral representation of the fractional Laplacian operator [19]. Using the method of series expansion, Ye et al. [20] studied the multi-term time-space fractional partial differential equations in 2D and 3D domains.

However, numerical methods on the multi-term time-fractional differential equations are still under development. Liu et al. [14] recently developed several finite difference schemes to the multi-term time fractional advection-dispersion and wave-diffusion equations based on the fractional predictor-corrector method (FPCM). In [21], the authors extended the predictor-corrector method combined with the L1 and L2 schemes to the multi-term time-space Riesz–Caputo fractional equations. As in the finite difference method for solving the single-term time-fractional diffusion equation, the convergence order of FPCM is $O(\tau^{1+\alpha_1})$. On the other hand, a high-order and effective numerical method is always necessary due to the non-locality of the fractional differential operator.

Recently, the authors have developed several high-order methods for solving the time fractional diffusion equations (see [22–24]). These methods are based on the Galerkin finite element/spectral approximation or space-time spectral approximation to attain high-order accuracy. It is well-known that the Galerkin spectral/finite element methods are superior to finite difference methods in many instances for the standard partial differential equations (see [25–28]). However, the extension of the Galerkin method to the fractional differential equation is not trivial (see [29,30]). The space of basis functions and corresponding approximation estimates require careful handling due to the singularity of the fractional differential operator. To date, application of the spectral/finite element methods to the fractional diffusion equations (see [31–33]) has yielded convergence rates of the order $O(\tau^{2-\alpha} + \tau^{-1}h^r)$ with time and space step sizes τ, h and regularity order r. Obviously, the convergence rates in these works are not optimal due to the presence of the factor τ^{-1}. In [23], the authors improved the order to $O(\tau^{2-\alpha} + h^{r+1})$ using the finite difference/spectral method, where r is the polynomial degree. In [22], the authors employed the space-time spectral method for solving the time fractional diffusion equation, and the spectral accuracies both in space and time were obtained, but only zero initial condition and homogeneous boundary conditions were considered. Compared to the Galerkin spectral/finite element methods, the space-time spectral method possesses much higher accuracy due to the exponential/algebraic convergence in both time and space directions.