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# A novel fractional grey system model and its application

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ABSTRACT

Of the grey models proposed for making predictions based on small sample data, the GM(1,1) model is the most important because of its low demands of data distribution, simple operation, and calculation requirements. However, the classical GM(1,1) model has two disadvantages: it cannot reflect the new information priority principle, and, if it is necessary to obtain the ideal effect of modeling, the original data must meet the class ratio test. This paper presents a new fractional grey model, FGM($q$, 1), which is an extension of the GM(1,1) model in that first-order differential equations are transformed into fractional differential equations. Decomposition of the data matrix parameters during the process of solution shows that the new model follows the new information priority principle. For modeling, the mean absolute percentage error (MAPE) is established as the objective function of the optimization model, and a particle swarm algorithm is used to calculate the accumulation number and the order of the differential equation that can minimize the MAPE. Finally, the results from three groups of data modeling show that, compared with other classical grey models, FGM($q$, 1) has higher modeling precision, can overcome the GM(1,1) model class ratio test restrictions and has a wider adaptability.



## 1. Introduction

Grey system theory was proposed by Deng [1,2] to solve problems for which the information is poor, incomplete, or uncertain. The GM(1,1) model, the foundation of this theory, was established based on the nature of differential properties for a discrete sequence of data of one variable. As shown in Fig. 1, the first “1” refers to a first-order differential equation, and the second “1” indicates that there is only one variable in the model. With the advantage of a simple modeling process and low demands regarding the distribution pattern of the data sequence, the GM(1,1) model has been widely applied to predictions of complex systems, such as vehicle fatality risk [3], Lorenz chaotic system [4], labor formation [5] the moving path of the typhoon [6] and energy consumption [7]. Although these successes have been achieved, to establish the satisfactory models still need some precondition. Ying [8] proposed the theory of the admissible region of class ratio, which states that for a satisfactory GM(1,1) model to be established, the class ratio of the original sequence should be in $(e^{-\frac{2}{n+1}},\ e^{\frac{2}{n+1}})$, where $n$ is the length of the original sequence. Deng [9] summarized three necessary condition about grey modeling and thought the theory of the admissible region of class ratio could help to judge the feasibility of GM(1,1) modeling.

The priority of new information is a significant principle in forecasting theory, which means that latest information has most reference value for modeling. However, the classical GM(1,1) model did not materialize this principle. Scholars have

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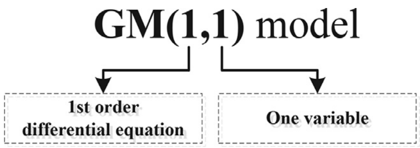


**Fig. 1.** GM(1,1) model.

found that improvement of the accumulated generating operation could emphasize it, such as the buffer operator proposed by Liu [10,11], improved buffer operator proposed by Dang [12,13] and Li [7]. Although emphasizing the principle of new information priority, the subjectivity limits application of these models when determining the weighting efficient in buffer operator. Expanding first accumulated generating operation into fractional accumulated generating operation, Wu proposed the fractional accumulation GM(1,1) model(the FAGM(1,1) model) and thought this model also embodied the new information priority [14]. Successfully applied into the maintenance cost of weapon system [15] and the gas emission [16], this model showed the remarkable predication. Xiao [17,18] regarded the fractional accumulated generating matrix as a specifical buffer operator, a kind of generalized accumulated generating. Mao [19] thought the fractional accumulated generating as a special data transformation, and he proved that the FAGM(1,1) could overcome the restrictions regarding class ratio of the GM(1,1) model.

The current fractional grey model just used fractional order accumulated generating, while the whitening differential equation of the model is still the classic integer-order differential equation not the fractional differential equation. Although this incomplete combination has obtained some achievement, there is still room for improvement in forecasting.

The idea that an integer-order derivative can be extended to a non-integer order can be traced to a communication between Leibniz and LHopital in the 1600s. Since the first monograph about fractional calculus was published in 1974 [20], fractional model has rapidly developed. The fractional differential equation(FDE) is the core of these models. Considering the admirable memory principle [21] in FDE, these models have been successfully applied into chemical processing systems [22], hard drive design [23], and denoising for texture images [24]. However, there is still not any rational and effective way to obtain analytic solution of general FDE. In fact, the FDE is usually deemed to be the limit form of fractional difference equation [25]. The numerical simulation of FDE could be obtain when FDE is transformed into fractional difference equation [20] (i.e., finite element method or finite difference method). Although the development in numerical solution boom the application of FDE, Although the development in numerical solution boom the application of FDE, the speed and accuracy usually cannot be both guaranteed in the same numerical techniques for the solution [26,27].

Considering the memory principle in FDE, based on the original GM(1,1) model combined with the fractional accumulated generating matrix, and further extending the whitening differential equation of the model to the fractional differential equation, this paper proposes the fractional grey model, FGM(q,1), which has a more complete combination and a wider range of applications. Though there exit some errors when adopt the numerical simulation, the variable and data in our model are pitifully small. Its effective to solve the equation by transforming FDE into difference equation system.

The remainder of this paper is organized as follows. In Section 2, the fractional accumulation and inverse matrices, which are based on matrix theory, are introduced, and the fractional grey model FGM($q$, 1), in which the differential equation is expanded from first order to fractional order, is proposed. In Section 3, we discuss the transformation of a fractional differential equation to a fractional difference equation to calculate predicted and reverted values via fractional accumulation matrices. In Section 4, two examples and a real-life case are used to compare the new FGM($q$, 1) model with the three previous models: GM(1,1), DGM(1,1), and FAGM(1,1).

## 2. Fractional accumulated generating matrices and their inverse matrices

Accumulated generating operation (AGO) is an important original component of grey system theory, the main purposes of which are to reduce the volatility of raw data and improve the grey exponential rate, the theoretical basis for modeling grey differential equations. If a single AGO is not enough, multiple operations can be performed, but using accumulated generating too many times may damage the grey exponential rate of a series [11,17]. Thus, determining the order of an accumulation matrix is very important for an ideal model.

For an original sequence $x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))^T$, a $r$th order accumulated generating operation($r$-AGO) can be defined as follows.

**Definition 1.** Let sequence $x^{(r)}$ be the $r$-AGO sequence of $x^{(0)}$, where $x^{(r)}(k) = \sum_{i=1}^{k} x^{(r-1)}(i), k = 1, 2, \ldots, n$. In particular, when $r = 1$, $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i)$, and the expression $x^{(1)}$=$\mathbf{A}x^{(0)}$ can be obtained from matrix operation theory, where $\mathbf{A}$ is a 1-AGO matrix, and

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$