Applied Mathematical Modelling 40 (2016) 5077–5095

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

# Dispersion analysis of discontinuous Galerkin method on triangular mesh for elastic wave equation

Vadim Lisitsa*

[a] Institute of Petroleum Geology and Geophysics SB RAS, Novosibirsk, Russia
[b] Novosibirsk State University, Novosibirsk, Russia

ARTICLE INFO

*Article history:*
Received 21 April 2014
Revised 17 August 2015
Accepted 11 December 2015
Available online 30 December 2015

*Keywords:*
Seismic waves
Discontinuous Galerkin
Finite differences
Elastic wave equation
Numerical dispersion

ABSTRACT

This paper presents a mid-frequency dispersion analysis of the triangle-based discontinuous Galerkin method for numerical simulation of seismic wave propagation. The results for different orders of basis polynomials are presented and compared with the finite difference approximations. It is shown that the dispersion error of the P1 DG is higher than that of the second order standard staggered grid scheme if the dispersion error is considered with respect to the degrees of freedom per wavelength, and the error of P2 is close to that of the fourth order scheme, whereas a further increase in the polynomial order makes it possible to simulate seismic waves using as coarse discretization as 2 grid cells per wavelength however computational intensity of the algorithm becomes unreasonably high.



## 1. Introduction

Modern computational resources and advanced numerical techniques make it possible to simulate wave propagation in complex realistic models. On the one hand, such simulations are used in academic application to study peculiarities of wave propagation in presence of anisotropy [1–3]; in presence of attenuation [4–8]; in presence of small-scale structures [9–13]; in presence of sharp interfaces and surface waves due the discontinuities of the model [14–18]; in composite materials [19] and others. On the other hand, simulation of seismic wave propagation is an essential part of seismic processing algorithms such as reverse time migration [20,21] and full waveform inversion [22,23].

Note that seismic modeling is typically based on the finite difference technique. This approach combines simplicity because regular hexahedral meshes are used; efficiency because explicit schemes are typically utilized; applicability to models of almost arbitrary complexity; ease of parallel implementation on the base of domain decomposition, and reliable accuracy [24]. Moreover, there is a variety of schemes specified for particular features of the models; for example, simulation of seismic wave propagation in isotropic media is typically performed by means of the standard staggered grid scheme [25,26]; in the presence of anisotropy Lebedev [3,27], or a rotated staggered grid [1] scheme is applied, etc. However, presence of sharp interfaces in the model may significantly reduce the accuracy of finite difference simulations [28–30] because of low quality stair-step approximation of the interface. To overcome this, a variety of approaches were suggested, starting with curvilinear a coordinate transform with further use of finite differences [14,15,18] and finishing with advanced numerical techniques such as finite elements [31], including spectral elements [32,33], finite volumes [34], discontinuous Galerkin (DG) methods

* Correspondence address: Institute of Petroleum Geology and Geophysics SB RAS, Novosibirsk, Russia. Tel.: +7 3833301337.
*E-mail address:* lisitsavv@ipgg.sbras.ru, vlisitsa@ngs.ru

http://dx.doi.org/10.1016/j.apm.2015.12.039

[35–44] and others. If elastic wave propagation is considered with application to engineering and, in particular, to composite material study, for example, the investigation of band gaps in functionally graded materials and phononic crystals finite differences may have one more drawback, i.e., if finite differences are applied to models with smoothly varying coefficients, these coefficients are typically substituted by suitable averaging [29,30,45] which leads to the second order of the solution convergence regardless of the formally high order of approximation of elastic wave equation by a scheme. In addition, for engineering purposes, a high quality solution almost free from numerical dispersion is required because the study of graded material properties is based on the dispersion of the solution and on the frequency dependency of the reflection/transmission coefficients. Therefore, in this area of research, semi-analytical approaches [46,47] and highly accurate formulation of finite element techniques [19] are in use.

In this paper, I focus on the DG mainly in application to seismic modeling problems, where typical discretizations vary between 3 points (grid cells) per wavelength to preserve the model resolution and 50 points per wavelength to keep the algorithm applicable to problems of realistic size. The DG is preferred for seismic modeling because of several reasons, such as relative simplicity of implementation, ease of parallel implementation via domain decomposition, and the possibility of constructing "almost" explicit time discretizations. In addition, DG allows hp-adaptivity; i.e., discontinuous locally refined meshes [11,13,48] can be used to match the fine structure of the model or, local choice of basis functions can be performed to improve accuracy [42]. In particular, the goal of the paper is to perform dispersion analysis of a discontinuous Galerkin method applied to velocity-stress (first order) formulation of elastic wave equation to estimate the numerical error due to particular discretization and to formulate the criterion on the discretization construction to achieve prescribed accuracy. This research has two main peculiarities. First, I focus on the regular triangular meshing of the domain. Regularity of the mesh is an essential part of seismic wave simulation, as the models are typically provided on some regular hexahedral (rectangular) mesh and only several sharpest interfaces (free-surface, bathymetry) are defined explicitly. This means that the triangulation is also regular with only local irregularities in the vicinity of the prescribed interfaces. Second, I am not about to discuss the asymptotic behavior of the numerical dispersion because it has been rather extensively discussed in a series of papers [49–55]. In addition, I am not going to discuss the high-frequency regimes (discretizations below 3–4 grid cells per wavelength) and the behavior of spurious parasitic modes either. These studies are presented in [50,56–59]. So, the objective of this research is a mid-frequency dispersion analysis of the DG for an elastic wave equation, which makes it possible to choose the grid step for practical applications. To establish a link between obtained results and those reported previously on the numerical dispersion of the other methods for seismic modeling, such as finite-differences [3,60,61], discontinuous Galerkin and the spectral elements method for rectangular meshes [44,62,63] I support my results with dispersion analysis of the commonly used standard staggered grid scheme of the second [25] and the fourth [26] order of approximation. To make the comparison consistent, the dispersion curves (dependence of the relative velocity error on the discretization) are presented in a standard way; i.e. with respect to the number of grid cells per wavelength, and in terms of degrees of freedom per wavelength.

The paper is organized as follows. In Section 2, I refer to the DG formulation for the elastic wave equation. Section 3 contains a detailed description of the dispersion analysis applied to the DG formulation. Next, the numerical study of the dispersion for the mid-frequency regimes for DG and finite differences are presented and compared in Section 4. The results are concluded in Section 5.

## 2. Preliminaries

Consider a hyperbolic system written in divergence-free form in 2D Cartesian coordinates:

$$
\begin{aligned}
&A_1 \frac{\partial \vec{u}}{\partial t} - \sum_{j=1}^{2} B_j \frac{\partial \vec{\sigma}}{\partial x_j} = 0,\\
&A_2 \frac{\partial \vec{\sigma}}{\partial t} - \sum_{j=1}^{2} B_j^* \frac{\partial \vec{u}}{\partial x_j} = 0,
\end{aligned} \tag{1}
$$

here, matrices $A_1$ and $A_2$ are self-adjoint and strictly positive definite, and matrices $B_1$ and $B_2$ are independent on the coordinates. This representation is valid for the acoustic wave equation [18], elastic wave equation [3,18], and Maxwell wave equation. In this paper, I will focus on the elastic wave equation where:

$$
A_1 = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad A_2 = S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix},
$$

$$
B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
$$

where $\rho$ is the mass density, $S$ is the compliance matrix; i.e., the inverse of stiffness matrix $C$. In these notations, $\vec{u} = (u_1, u_2)^T$ is the particle velocity vector and $\vec{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{12})^T$ is the stress tensor, written as a vector [64]. Below, only the