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Quasi-reversibility method to identify a space-dependent source for the time-fractional diffusion equation

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ABSTRACT

In this study, we consider a problem of recovering a space-dependent source for the time-fractional diffusion equation, where the additional data is the observation at a final moment $t = T$. We develop a quasi-reversibility method to overcome the ill-posedness of the problem. The convergence estimates under an a priori parameter choice rule and an a posteriori parameter choice rule are proved, respectively. Finally, numerical examples are presented which demonstrate the effectiveness of the regularization methods and confirm the theoretical claims.

1. Introduction

In this paper, we are concerned with an inverse source problem for the time-fractional diffusion equation with variable coefficients in a general bounded domain.

Let Ω be a bounded domain in $\mathbb{R}^d (1 \leqslant d \leqslant 3)$, with sufficient smooth boundary $\partial\Omega$. $L^2(\Omega)$ is a usual L^2 space with the scalar product $(\cdot,\cdot)$, and $H^l(\Omega), H_0^l(\Omega)$ denote usual Sobolev spaces (e.g., Adams [1]). We consider a partial differential equation with the Caputo fractional derivative in time t as follows:

$$\begin{cases} D_t^\alpha u(x,t) = (Lu)(x,t) + f(x), & x \in \Omega,\ t \in (0,T),\ 0 < \alpha < 1, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \\ u(x,0) = 0, & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where D_t^α is the Caputo fractional derivative of order $\alpha (0 < \alpha \leqslant 1)$ defined by

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha} d\tau, & 0 < \alpha < 1, \\ u_t(x,t), & \alpha = 1; \end{cases} \tag{1.2}$$

and $-L$ is a symmetric uniformly elliptic operator defined on $D(-L) = H^2(\Omega) \bigcap H_0^1(\Omega)$ given by

$$Lu(x) = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(\sum_{j=1}^d a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) + c(x)u(x), \quad x \in \Omega, \tag{1.3}$$

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in which the coefficients satisfy

$$a_{ij} = a_{ji}, \quad 1 \leqslant i, j \leqslant d, \ a_{ij} \in C^1(\overline{\Omega}), \tag{1.4}$$

$$\nu \sum_{i=1}^{d} \xi_i^2 \leqslant \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j, \quad x \in \overline{\Omega}, \ \xi \in \mathbb{R}^d, \text{ for a constant} \nu > 0, \tag{1.5}$$

$$c(x) \leqslant 0, \quad x \in \overline{\Omega}, \ c(x) \in C(\overline{\Omega}). \tag{1.6}$$

The fractional diffusion equations have attracted wide attentions in recent years and it can model anomalous diffusion phenomena, where a classical diffusion advection equation does not simulate real field data [2–4]. However, for some practical problems, the initial data, or part of boundary data, or diffusion coefficients, or source term may not be given and we want to recover them by additional measurement data which will yield to some fractional diffusion inverse problems [5–18].

In this study, we want to recover the source function $f(x)$ from indirect observable data at a final moment $t = T$. The additional data contain measurement errors and satisfies

$$\|g^\delta(x) - u(x, T)\| \leqslant \delta, \tag{1.7}$$

unless otherwise specified, in this paper, $\| \cdot \|$ refers to the L^2 norm and $\delta > 0$ is a noise level.

For $\alpha = 1$, it is a classical ill-posed problem and has been studied by many researchers. We refer the reader to [19] and the references therein. Recently, a regularization method based on the idea of quasi-reversibility method [20] has been used to deal with this problem. But the authors only considered the problem in one dimensional case and gave the error estimate under the a priori parameter choice rule. For $\alpha < 1$, Liu et al. in [8] used a quasi-reversibility method to solve the backward problem in one-dimensional case for special coefficients and Sakamoto et al. gave the existence and uniqueness results in [14]. Wang et al. [19,21] employed the Tikhonov regularization method and the quasi-boundary value method to deal with the backward time-fractional diffusion problem. For the inverse source problem, Wang et al. [22] used Tikhonov regularization method and a simplified Tikhonov regularization method to solve it in one-dimensional case and gave the convergence estimates.

In this study, we propose a quasi-reversibility method to solve this inverse source problem for fractional diffusion equation with variable coefficients in a general bounded domain. We give the convergence rate under an a priori bound assumption of the exact solution and a an priori parameter choice rule. We also provide an a posteriori parameter choice rule and give the corresponding convergence rate estimate. To our knowledge, To our knowledge, the proposed a posteriori parameter choice rule is a new development even in the integer-order case. All the numerical results are based on the a posteriori parameter choice rule which is independent of the a priori bound E of the exact solution. It is more useful in practical situations.

The outline of the paper is as follows: some preliminary results are introduced in Section 2. The ill-posedness and the conditional stability of the inverse problem (1.1) are discussed in Section 3. In Section 4, we constructed the regularized solution by a quasi-reversibility method. Moreover, the detailed convergence analysis under an a priori regularization parameter choice rule and an a posteriori regularization parameter choice rule are carried out. Finally, some numerical examples are given in Section 5 to confirm theoretical analysis.

2. Preliminaries

Throughout this paper, we use the following definition and lemmas.

Definition 2.1 *[23]*. The Mittag–Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in R$ are arbitrary constants.

Lemma 2.1 *[24]*.

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}. \tag{2.8}$$

Lemma 2.2 *[23]*. *Let* $\lambda > 0$, *then*

$$\int_0^\infty e^{-pt} t^{\gamma k + \beta - 1} E_{\gamma,\beta}^{(k)}(\pm a t^\gamma) dt = \frac{k! p^{\gamma-\beta}}{(p^\gamma \mp a)^{k+1}}, \quad Re(p) > |a|^{\frac{1}{\gamma}}, \tag{2.9}$$