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Formulation of a geometrically nonlinear 3D beam finite element based on kinematic-group approach

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ABSTRACT

A relatively simple finite element with 12 degrees of freedom is proposed for geometrically nonlinear analysis of spatial shear deformable beams and rods. The finite-element formulation is based on the concept of kinematic group, which is a geometrical object comprising two nodes on the rod axis and two adjoined vectors (directors) which define orientation of the cross section at each node. Results of sample problems are given to show the applicability of the element to study nonlinear deformation and stability of three-dimensional elastic rods undergoing finite rotations.

1. Introduction

Flexible rods represent a special class of thin elastic structures. In contrast to plates and shells, thin rods can easily be deformed such that substantial changes in the equilibrium configurations occur without exceeding the elastic limit of the material. Investigation of equilibrium configurations and stability of flexible beams and rods under general loading conditions poses a challenging topic in the nonlinear theory of elasticity.

The history of mathematical modeling and simulation of elastic flexible rods dates back to the planar elastica problem studied by James and Daniel Bernoulli and Euler [1,2]. Three-dimensional bending curves as a generalization of Euler's elastica were studied by Kirchhoff who found an analogy between mathematical description of the curves and that of a rigid body rotating about a fixed point. Nonlinear equations governing three-dimensional equilibrium states of rods are very complicated and, in general case, strongly resist analytical solution. Exact solutions are available only for particular cases of simple loading and boundary conditions; therefore, in practice, one has to resort to numerical techniques. In the 1960s, the finite element method emerged as a powerful tool for solving various engineering problems and much effort has been spent to develop effective elements for geometrically nonlinear analysis of 3D flexible rods ever since.

It is well known that the finite element of a deformable solid must be capable of reproducing all rigid body motions to ensure convergence of the solution with the mesh refinement. This means that the strains in the element must vanish under rigid body motion. The question of choice of functions interpolating unknown variable fields in the element is of crucial importance to satisfy the criterion. The main difficulty that complicates the construction of nonlinear finite elements of rods is that standard interpolation techniques are not applicable because of non-vectorial nature of finite rotations in 3D space.

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To overcome this difficulty, many studies have exploited the fact that displacement fields of the element can be decomposed into rigid body motion and pure deformation modes. In a series of papers by Besseling [3–5], the initially straight 2-node beam element was formulated using six deformation parameters or generalized strains defined in a local coordinate system and expressed in terms of 12 nodal parameters by complicated nonlinear relations. It was shown that the generalized strains possess the invariance with respect to rigid body motion of the element. Argyris and co-workers [6,7] proposed a 3D beam formulation based on the natural deformation modes which describe small local deformations of the element. Separated from large rigid body rotations, the natural modes are interpolated along the beam length using simple strain–displacement relations. The geometrical nonlinearity is taken into account by nodal displacements and rotations in the global coordinate system. To avoid difficulties associated with non-commutativity of successive finite rotations, they introduced the concept of semitangential rotations, which leads to symmetric geometric stiffness matrix of the element. Based on the Besseling approach, Jonker and Meijaard [8] have recently proposed 3D beam element with discrete deformation modes which take into account nonlinear second-order terms in the strain–displacement relations.

The idea of decomposition of the total deformation into rigid body motion and pure deformation modes has been implemented in the co-rotational formulations where local reference frame is introduced for each element to capture deformation response. The reference or "ghost" frame is continuously translates and rotates with the element but does not deform. Provided the element is small, its deformations measured with respect to co-rotated position are small; therefore, it is possible to employ linear or simplified nonlinear strain–displacement relations in the local reference frame. The geometric nonlinearity is incorporated in the transformation matrices relating local and global quantities. Co-rotational formulations of three-dimensional beam elements were proposed by Pacoste and Ericsson [9], Hsiao et al. [10,11], Crisfield [12], and Battini and Pacoste [13] to mention just a few.

An original rotationless formulation was proposed by Shabana [14] to avoid difficulties arising in the interpolation of finite rotations along the element. Absolute nodal position and slope degrees of freedom are used to approximate the position field of the element. However, this approach results in a large number of degrees of freedom compared to conventional beam elements.

In the present paper, a relatively simple finite-element formulation is proposed for nonlinear analysis of three-dimensional rods undergoing large displacements and rotations but small strains. The formulation is based on the concept of kinematic group (KG) defined as a geometrical object that comprises two nodes on the rod axis and two adjoined vectors (directors) at each node. As a mechanical object, the group can perform rigid-body motions in 3D space and undergo changes in the configuration. Since the rigid-body modes are taken into account at the KG level, the issue of finite-element interpolation reduces to expressing the strains in the element in terms of changes in the configuration of the kinematic group. The latter are determined by scalar products of the nodal vectors. Defined in this manner, the strains in the element are invariant with respect to rigid body motions.

The paper is set out as follows. Section 2 summarizes simplifying assumptions used to construct the rod finite element. In Section 3, the concept of kinematic group of a 3D rod and its strain measures are discussed. In Sections 4 and 5, strains and strain energy of the rod finite element are expressed in terms of the kinematic group strains. In Section 6, the first and second variations of the strain energy are obtained. In Sections 7 and 8, numerical algorithm for determining the equilibrium states and investigating their stability is outlined. Section 9 contains numerical examples to demonstrate accuracy, shear locking-free behavior and nonlinear capabilities of the finite element.

2. Basic assumptions

To construct the finite element of a spatial rod, we adopt the following assumptions:

(a) The material is linear elastic and isotropic;
(b) The strains are small compared to unity, but arbitrarily large displacements and rotations are allowed;
(c) The shear center and centroid of the cross section coincide;
(d) Initial twist of the rod is equal to zero;
(e) The effect of constrained rotation and warping of the cross sections are ignored;
(f) Transverse shear deformations are taken into account using the Timoshenko hypothesis: the cross sections plane and normal to the rod axis in the initial state remain plane but not necessarily normal to the deformed axis.

3. Kinematic group of the rod

We consider the rod axis as a spatial curve that passes through centroids of the cross sections. Equilibrium configurations of the rod can be determined by the position of centroids and orientations of the cross sections. We define two nodes on the rod axis whose position is determined by the vectors $\boldsymbol{R}_0$ and $\boldsymbol{R}_1$. At each node, we introduce two unit vectors (directors) $\boldsymbol{d}_{mp}$ which lie in the cross-sectional plane (the subscript p enumerates directors and runs from 1 to 2 and subscript m enumerates nodes and takes the values 0 or 1). In Cartesian frame of reference with the base unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$, the nodal vectors are decomposed as

$$\boldsymbol{R}_m = \boldsymbol{e}_i x_{mi}, \ \boldsymbol{d}_{mp} = \boldsymbol{e}_i \lambda_{mpi} \quad (i = 1, 2, 3; m = 0, 1; p = 1, 2), \tag{3.1}$$