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A note on mathematical treatment of the Dirac-delta function in the differential quadrature bending and forced vibration analysis of beams and rectangular plates subjected to concentrated loads

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ABSTRACT

There are many physical and mechanical phenomena that can be well described by means of the Dirac-delta function. For instance, the bending and vibration behavior of structures under concentrated loads, impulsive loading, moving loads, and impact loading; and thermoelastic vibration behavior of structures under heat source points can be mathematically modeled by means of the Dirac-delta function. It is well known that such phenomena or problems can be easily handled by weak form based methods such as the Ritz and finite element methods. However, the strong form based methods such as the finite difference and differential quadrature methods may encounter some difficulties in mathematical modeling and treatment of the Dirac-delta function. This is mainly caused by the fact that the properties of the Dirac-delta function are in the form of integrals and not in the form of derivatives. To overcome this difficulty, this paper presents a combined differential quadrature–integral quadrature method in which such type of problems can be easily and accurately modeled. Its accuracy and reliability are demonstrated through the static and dynamic analysis of beams and rectangular plates under concentrated loads. This paper also presents a simple differential quadrature formulation for the analysis of rectangular plates. The proposed formulation first reduces the original plate problem to two simple beam problems. Each beam problem in then discretized using the differential quadrature method (DQM) in a simple manner. Compared with the conventional DQM, the proposed DQM is superior since its implementation and programming are easier and simpler.

1. Introduction

The Dirac-delta function, which was first introduced by theoretical physicist Paul Dirac [1] in 1958, is a generalized singularity function that has zero value everywhere except one point, with an integral of one over the entire domain. For example, the one dimensional Dirac-delta function has the following properties:

$$\delta(x - x_0) = 0 \text{ for all } x \neq x_0, \tag{1}$$

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$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1, \tag{2}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0) dx = f(x_0), \tag{3}$$

where $f(x)$ may be any arbitrary function. It is noted that the one dimensional Dirac-delta function is sometimes assumed to have infinite value at $x = x_0$. For this reason, the one dimensional Dirac-delta function exhibits strong singular behavior at $x = x_0$, and thus it is also called singularity function.

Similarly, the two dimensional Dirac-delta function is defined as:

$$\delta(x - x_0, y - y_0) = \delta(x - x_0)\delta(y - y_0) = 0 \text{ for all } x \neq x_0 \text{ and } y \neq y_0, \tag{4}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0)\delta(y - y_0) dxdy = 1, \tag{5}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x - x_0, y - y_0) dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x - x_0)\delta(y - y_0) dxdy = f(x_0, y_0), \tag{6}$$

where $f(x,y)$ may be any arbitrary function. It can be seen from Eqs. (1)–(6) that two properties of the Dirac-delta function are in the form of integrals.

Due to above particular features, the Dirac-delta function has been used in many areas of mathematics, physics and engineering. In the engineering area, for instance, the behavior of structures under concentrated loads, impulsive loading, impact loading, or heat/pressure point source can be mathematically modeled by means of the Dirac-delta function. In general, such problems can be handled using analytical and/or numerical approaches. In order to solve such type of problems numerically, one may use the weak form based methods or strong form based methods. The weak form based methods such as the finite element method can easily handle such type of problems since they directly integrate the governing equation of the problem [2]. However, the strong form based methods such as the differential quadrature method (DQM) and the discrete singular convolution method (DSCM) may encounter some difficulties when dealing with such type of problems [3,4]. For example, Zhang and Zhong [3] reported that the relative error of the center deflection by the conventional DQM can be as high as 20% even with 21 grid points for an isotropic simply supported square plate. The reason for this is that the DQM directly satisfies the governing equation of the problem at some grid points and approximates the derivatives numerically. Since the properties of the Dirac-delta function are in the form of integrals, the conventional DQM cannot handle such type of problems. To overcome this difficulty, Wang et al. [4] proposed the following approximation for the two dimensional Dirac-delta function:

$$\delta(x - x_0, y - y_0) = \delta(x - x_0)\delta(y - y_0) = \begin{cases} 1/(\bar{\Delta}x\bar{\Delta}y) & (x,y) = (x_0,y_0) \\ 0 & (x,y) \neq (x_0,y_0) \end{cases}, \tag{7}$$

where $\bar{\Delta}x$ and $\bar{\Delta}y$ are the average grid spacing in the x- and y-directions, respectively. Noting that the two dimensional Dirac-delta function should give infinite value at $x = x_0$ any $y = y_0$, it can be seen from Eq. (7) that the approach of Wang et al. [4] can produce highly accurate solution only when:

$$\bar{\Delta}x \to 0 \text{ and } \bar{\Delta}y \to 0, \tag{8}$$

which means very small grid spacing should be used in the method. Therefore, a large number of grid points need to be used in this approach which can then increase the computational cost of the method.

To overcome the above-mentioned difficulties, this paper presents a simple mixed differential quadrature-integral quadrature method in which the Dirac-delta function is simply and accurately approximated. For example, in the case of one dimensional problems involving the Dirac-delta function, the DQM is first used to discretize the derivatives at all the domain except at the grid point $x = x_0$. The Integral quadrature method (IQM) is then applied to treat the Dirac-delta function mathematically and to obtain the discrete equation of the problem at the grid point $x = x_0$. The accuracy and reliability of the proposed approach are demonstrated through the static and dynamic analysis of beams and rectangular plates under concentrated loads. It is shown that the proposed mixed approach produces much better accuracy than the Wang approach using a smaller number of grid points.

2. Differential quadrature method (DQM)

The DQM, which was first introduced by Bellman and his associates [5,6] in the early 1970s, is a powerful point discretization method for the numerical solution of partial differential equations appearing in science and engineering. It was applied for the first time to a structural mechanics problem by Bert et al. [7,8], and Jang et al. [9]. They solved the bending and vibration problems of beams and rectangular plates with various boundary conditions. Since then, the DQM has been successfully applied to a variety of engineering problems [10–26]. Most of these applications are related to static and dynamic analysis of structural components like beams, plates, and shells. Newer applications include the use of the DQM for approximation of