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An inverse time-dependent source problem for the heat equation with a non-classical boundary condition

A. Hazanee [a], D. Lesnic [a,*], M.I. Ismailov [b], N.B. Kerimov [c]

[a] Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK
[b] Department of Mathematics, Gebze Institute of Technology, Gebze, Kocaeli 41400, Turkey
[c] Department of Mathematics, Mersin University, Mersin 33343, Turkey

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ABSTRACT

This paper investigates the inverse problem of determining the time-dependent heat source and the temperature for the heat equation with a non-classical boundary and an integral over-determination conditions. The existence, uniqueness and continuous dependence upon the data of the classical solution of the inverse problem is shown by using the generalised Fourier method. Furthermore in the numerical process, the boundary element method (BEM) together with the second-order Tikhonov regularization is employed with the choice of regularization parameter based on the generalised cross-validation (GCV) criterion. Numerical results are presented and discussed.

1. Introduction

Inverse time-dependent source problems for the heat equation with local, nonlocal, integral or nonclassical (boundary) conditions have become the point of interest in many recent papers, [1–7], to name only a few. In the present paper, we consider yet another reconstruction of a time-dependent heat source from an integral over-determination measurement of the thermal energy of the system and a new dynamic-type boundary condition.

Let $T > 0$ be a fixed number and denote by $D_T = \{(x,t) : 0 < x < 1, 0 < t \leqslant T\} = (0,1) \times (0,T]$. Consider the following initial-boundary value problem for the heat equation:

$$u_t = u_{xx} + r(t)f(x,t), \quad (x,t) \in \overline{D}_T, \tag{1.1}$$

$$u(x,0) = \varphi(x), \quad x \in (0,1), \tag{1.2}$$

$$u(0,t) = 0, \quad t \in (0,T], \tag{1.3}$$

$$au_{xx}(1,t) + du_x(1,t) + bu(1,t) = 0, \quad t \in (0,T], \tag{1.4}$$

where f, φ are given functions and a, b, d are given numbers not simultaneously equal to zero. When the function $r(t)$ is given, the problem of finding $u(x,t)$ from the heat Eq. (1.1), initial condition (1.2), and boundary conditions (1.3) and (1.4) is termed as the direct (or forward) problem. The well-posedness of this direct problem has been established elsewhere, [8].

* Corresponding author.
E-mail addresses: mmah@leeds.ac.uk (A. Hazanee), amt5ld@maths.leeds.ac.uk (D. Lesnic), mismailov@gyte.edu.tr (M.I. Ismailov), nazimkerimov@yahoo.com (N.B. Kerimov).

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This model can be used in heat transfer and diffusion processes with a source parameter present in (1.1). Also, in acoustic scattering or damage corrosion the dynamic boundary condition (1.4) is also known as a generalised impedance boundary condition, [9–12].

Taking into account the Eq. (1.1) at $x = 1$, the boundary condition (1.4) becomes

$$au_t(1,t) + du_x(1,t) + bu(1,t) = ar(t)f(1,t), \quad t \in (0,T]. \tag{1.5}$$

In order to add further physics to the problem, we mention that the boundary condition (1.5) is observed in the process of cooling of a thin solid bar one end of which is placed in contact with a fluid [13]. Another possible application of such type of boundary condition is announced in [14, p. 79], as this boundary condition represents a boundary reaction in diffusion of chemical. We finally mention that we have also previously encountered the dynamic boundary condition (1.5) when modelling a transient flow pump experiment in a porous medium [15].

When the function $r(t)$ for $t \in [0,T]$ is unknown, the inverse problem formulates as that of finding a pair of functions $(r(t), u(x,t))$ which satisfy the Eq. (1.1), initial condition (1.2), the boundary conditions (1.3) and (1.4) (or (1.5)), and the energy/mass overdetermination measurement

$$\int_0^1 u(x,t)\,dx = E(t), \quad t \in [0,T]. \tag{1.6}$$

It is also worth mentioning that a related parabolic inverse source problem given by equations (1.1)–(1.3), (1.6) and the following dynamic boundary condition

$$u_t(1,t) + u_x(1,t) + \sigma(u(1,t)) = 0, \quad t \in (0,T], \tag{1.7}$$

where σ is a given Lipschitz function, has very recently been investigated in [16]. However, no numerical results were presented and the boundary condition (1.7) is different of the boundary condition (1.5) considered in the present study.

The condition (1.6) is encountered in modelling applications related to particle diffusion in turbulent plasma, as well as in heat conduction problems in which the law of variation $E(t)$ of the total energy of heat in a rod is given, [17].

If we let $u(x,t)$ represent the temperature distribution, then the above-mentioned inverse problem can be regarded as a source control problem. The source control parameter $r(t)$ needs to be determined from the measurement of the thermal energy $E(t)$.

Because the function r is space independent, a, b and d are constants and the boundary conditions are linear and homogeneous, the method of separation of variables is suitable for studying the problem under consideration. It is well-known that the main difficulty in applying the Fourier method is the explicit availability of a basis, i.e. the expansion in terms of eigenfunctions of the auxiliary spectral problem

$$\begin{cases} y''(x) + \mu y(x) = 0, \quad x \in [0,1], \\ y(0) = 0, \\ (a\mu - b)y(1) = dy'(1). \end{cases} \tag{1.8}$$

In contrast to the classical Sturm–Liouville problem, this problem has the spectral parameter also in the boundary condition. It makes it impossible to apply the classical results on expansion in terms of eigenfunctions [18]. The spectral analysis of such type of problems was started by [19]. After that, important developments were made by [20–24]. It is useful to note the reference [25] whose results on expansion in term of eigenfunctions will be used in the present paper.

The paper is organised as follows. In Section 2, the eigenvalues and eigenfunctions of the auxiliary spectral problem and some of their properties are introduced. Then the existence, uniqueness, and continuous dependence upon the data of the solution of the inverse problem (1.1)–(1.3), (1.5), (1.6) are proved. The numerical discretisation of the inverse problem is based on the boundary element method (BEM) which is described in Section 3. Section 4 discusses numerical results obtained for a couple of benchmark test examples and emphasises the importance of employing regularization in order to achieve a stable numerical solution. Finally, Section 5 presents the conclusions of the paper.

2. Mathematical analysis

Consider the spectral problem (1.8) with $ad > 0$. It is known from [20] that its eigenvalues $\mu_n, n = 0, 1, 2, \ldots$ are real and simple. They form an unbounded increasing sequence and the eigenfunction $y_n(x)$ corresponding to μ_n has exactly n simple zeros in the interval $(0,1)$. We can also give the sign of the first eigenvalue μ_0 as

$$\begin{cases} \mu_0 < 0 < \mu_1 < \mu_2 < \cdots, \text{if } -\frac{b}{d} > 1, \\ \mu_0 = 0 < \mu_1 < \mu_2 < \cdots, \text{if } -\frac{b}{d} = 1, \\ 0 < \mu_0 < \mu_1 < \mu_2 < \cdots, \text{if } -\frac{b}{d} < 1. \end{cases}$$

It was shown in [25] that the eigenvalues and eigenfunctions have the following asymptotic behaviour: