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Approximate solution of fuzzy differential equations under generalized differentiability

Maryam Mosleh [1], Mahmood Otadi *

Department of Mathematics, Firoozkooh Branch, Islamic Azad University, Firoozkooh, Iran

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ABSTRACT

In this paper, an approach for approximating the fuzzy linear system of differential equations under generalized differentiability is presented, because under this interpretation, we may obtain solutions which have a decreasing length of their support. In several applications the behavior of these solutions better reflects the behavior of some real-world systems. Also, we investigate a necessary and sufficient conditions for the solution vector to be a fuzzy vector.

1. Introduction

Usage of fuzzy differential equations is a natural way to model dynamical systems under possibilistic uncertainty [1]. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations which may appear in many applications. However the form of such an equation is very simple, it raises many problems since under different fuzzy differential equation concepts, the behavior of the solutions is different [2].

In modeling real systems one can be frequently confronted with a differential equation

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

where the structure of the equation is known, represented by the vector field *f*, but the model parameters and the initial value x_0 are not known exactly. One method of treating this uncertainty is to use a fuzzy set theory formulation of problem [3]. The topic of FDEs has been rapidly growing in recent years [4,5]. The concept of fuzzy derivative was first introduced by Chang and Zadeh [6], it was followed up by Dubois and Prade [7] who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [8] and Goetschel and Voxman [9]. Fuzzy differential equations were first formulated by Kaleva [10] and Seikkala [3] in time dependent form. Kaleva had formulated fuzzy differential equations, in term of Hukuhara derivative [10]. Buckley and Feuring [11] have given a very general formulation of a fuzzy first-order initial value problem. They first find the crisp solution, fuzzify it and then check to see if it satisfies the FDE also Khastan et al. [12] used variation of constant formula for first order fuzzy differential equations. Ming et al. [13] have a numerical solution of fuzzy differential equations base on the classical Euler method also Allahviranloo [14,15] have a numerical solution of fuzzy differential equations by predictor-corrector method. FDEs under generalized differentiability is considered by Nieto et al. [16]. One of the class of differential equations is the fuzzy linear system of differential equations that they have been

* Corresponding author. Tel.: +98 912 6964202.
E-mail addresses: mosleh@iaufb.ac.ir (M. Mosleh), otadi@iaufb.ac.ir (M. Otadi).
[1] Tel.: +98 912 6076308.

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discussed by [17]. Pearson [18] has a property of linear FDEs under H-differentiability, $\dot{x}(t) = Ax(t),\ x(0) = x_0$. Mosleh [19,20] used a novel hybrid method based on learning algorithm of fuzzy neural network for approximating the solution of system of differential equation with fuzzy initial values and fuzzy linear Fredholm integro-differential equation, then Mosleh and Otadi [21] used a fuzzy neural network for approximating the solution of fuzzy differential equation. Recently Mosleh and Otadi [22] have a new approach to solve the fuzzy linear system of differential equations based on pseudo inverse. This approach has the disadvantage that it leads to solutions which have an increasing length of their support [23]. The method provided in [22] gives a solution of a linear FDEs (or even system), but in the present paper we investigate the existence of the "other solutions" (local existence of two solutions is possible under the generalized differentiability concept according to [24]).

After a preliminary section we study fuzzy linear system of differential equations under generalized differentiability. Then we also provide some examples.

2. Preliminaries

Parametric form of an arbitrary fuzzy number is given in [25] as follows. A fuzzy number u in parametric form is a pair $(\underline{u}, \bar{u})$ of functions $\underline{u}(r),\ \bar{u}(r),\ 0 \leqslant r \leqslant 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,
2. $\bar{u}(r)$ is a bounded left continuous non-increasing function over $[0,1]$,
3. $\underline{u}(r) \leqslant \bar{u}(r),\ 0 \leqslant r \leqslant 1$.

The set of all these fuzzy numbers is denoted by E which is a complete metric space with Hausdorff distance. A crisp number α is simply represented by $\underline{u}(r) = \bar{u}(r) = \alpha,\ 0 \leqslant r \leq 1$.

For arbitrary fuzzy numbers $x = (\underline{x}(r),\ \bar{x}(r)),\ y = (\underline{y}(r),\ \bar{y}(r))$ and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as [25]

(a) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\bar{x}(r) = \bar{y}(r)$,
(b) $x + y = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$,
(c) $kx = \begin{cases} (k\underline{x}, k\bar{x}), & k \geqslant 0, \\ (k\bar{x}, k\underline{x}), & k < 0. \end{cases}$.

Definition 1 [26]. For arbitrary fuzzy numbers $u = (\underline{u}, \bar{u})$ and $u = (\underline{v}, \bar{v})$ the quantity

$$D(u, v) = \sup_{0 \leqslant r \leqslant 1} \{max[|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|]\}$$

is the Hausdorff distance between u and v. Puri and Ralescu in [8] introduced H-derivative (differentiability in the sense of Hukuhara) for fuzzy mappings and it is based on the H- difference of sets, as follows. Henceforth, we suppose $T = [t_0, t_0 + a]$ with $a > 0$.

Definition 2. A mapping $F : T \to E$ is differentiable at $t \in T$ if there exists a $\dot{F}(t) \in E$ such that the limits

$$lim_{h \to 0^+} \frac{F(t+h) \ominus F(t)}{h} \text{ and } lim_{h \to 0^+} \frac{F(t) \ominus F(t-h)}{h}$$

exist and are equal to $\dot{F}(t)$.

The above definition is a straightforward generalization of the Hukuhara differentiable function has increasing length of support. Note that this definition of derivative is very restrictive, the authors [24] introduced a more general definition of derivative for fuzzy number valued function. In this paper, we consider the following definition [27].

Definition 3. Let $F : T \to E$ and fix $t \in T$. One says F is (1)-differentiable at t, if there exists an element $\dot{F}(t) \in E$ such that for all $h > 0$ sufficiently near to 0, there exist $F(t+h) \ominus F(t),\ F(t) \ominus F(t-h)$ and the limits (in the metric D)

$$lim_{h \to 0^+} \frac{F(t+h) \ominus F(t)}{h} = lim_{h \to 0^+} \frac{F(t) \ominus F(t-h)}{h} = \dot{F}(t). \quad (1)$$

F is (2)-differentiable if for all $h < 0$ sufficiently near to 0, there exist $F(t+h) \ominus F(t), F(t) \ominus F(t-h)$ and the limits (in the metric D)

$$lim_{h \to 0^-} \frac{F(t+h) \ominus F(t)}{h} = lim_{h \to 0^-} \frac{F(t) \ominus F(t-h)}{h} = \dot{F}(t). \quad (2)$$