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Fuzzy extension of a classical function: An alternative to the knowledge base for modelling imprecise relations

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ABSTRACT

The aim of this paper is to introduce both the concept of fuzzy extension of a classical function for modelling imprecise relations between variables and the basic arithmetic operations which this entails.

The concept of fuzzy extension can be considered as a generalisation of the concept of classical function extended to the field of the fuzzy sets defined in R. The fuzzy extension $\bar{f}$ of a classical continuous function $f(x)$ is a particular kind of fuzzy relation, which describes the correspondence between two variables x and y. The univocal image of each value of x through $\bar{f}$ is a closed interval of y values $[f_l(x), f_u(x)]$. Functions $f_l(x)$ and $f_u(x)$ set the limits of the y interval whose extent of correspondence to x is nonzero, being zero $\forall y : y \leqslant f_l(x)$ and $\forall y : y \geqslant f_u(x)$ and they fulfill the condition that $\forall x \in D f_l(x) \leqslant f(x) \leqslant f_u(x)$. A fuzzy extension $\bar{f}$ is defined by its membership function $\mu_{\bar{f}}(x, y)$, which quantifies the extent to which each value of the y variable corresponds to each value of x. The image $\bar{y}$ of each fuzzy set $\bar{x}$ is achieved by means of the composition rule of $\bar{f}$. In order to perform arithmetic operations with fuzzy extensions, the following basic operations are defined: the addition $\bar{a}$, subtraction $\bar{s}$, multiplication $\bar{m}$ and division $\bar{d}$ of two fuzzy extensions. The concept of fuzzy extension of a classical function, the procedure to specify and model $\mu_{\bar{f}}(x, y)$, the procedure to determine the image $\bar{y}$ and the arithmetic operations are graphically illustrated and properly exemplified.

A fuzzy extension $\bar{f}$ defined in this way is a useful alternative to the set of fuzzy rules or knowledge base for modelling inherently imprecise relations, which are frequently described in a simplified manner by means of classical functions. For this reason, fuzzy extensions can be applied to a number of different fields, being particularly suitable for environmental assessments, such as the design and evaluation of environmental quality indexes and the environmental impact assessments, among others.

1. Introduction

Frequently, knowledge about the correspondence between two variables x and y by means of a function f in R is not accurate enough as to assign a unique value of y to each value of x. This is the case when there is a lack of information about the very correspondence between the variables, and/or when this information is affected by imprecision and uncertainty and/or

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when one or more of the related variables are not quantitative but linguistic. In such cases, fuzzy logic together with fuzzy set theory are efficient mathematical tools for the rigorous modelling of these imprecise relations, sets of fuzzy rules of the type IF/THEN being frequently used for this purpose [1].

The description of imprecise relations by means of sets of fuzzy rules has been widely applied to different fields in the industrial sector, such as business management [2,3], industrial manufacturing [4–6] and process control [7–9], among others. It has also been successfully used in medicine as a useful tool for the diagnosis of diseases [10–13]. Sets of fuzzy rules are particularly useful in the environmental field, where the correlations between the variables are very frequently inherently imprecise. Thus, they have been applied to decision support systems [14,15], quality assessments [16,17], environmental quality indexes [18–20] and environmental impact assessments [21,22].

The aim of this work is to introduce a new kind of fuzzy relation for the modelling of inherently imprecise relations, i.e. when the imprecision lies in the correspondence itself, regardless of the vagueness and/or inaccuracy which affect the values of the correlated variables. This new fuzzy relation is called *fuzzy extension of a classical function*. For this, we present and develop both the concept of fuzzy extension and the basic arithmetic operations, which are properly exemplified and graphically illustrated.

The concept of a fuzzy extension can be considered as a generalisation of the concept of a classical function extended to the field of the fuzzy sets defined in R, which describes the correspondence between two variables x and y. The univocal image of each value of x through $\bar{f}$ is a closed interval of y values $[f_l(x), f_u(x)]$. Functions $f_l(x)$ and $f_u(x)$ set the limits of the y interval whose extent of correspondence to x is nonzero. $\bar{f}$ is defined by its membership function $\mu_{\bar{f}}(x,y)$, which quantifies the extent to which each value of the y variable corresponds to each value of x. The image $\bar{y}$ of each fuzzy set $\bar{x}$ is achieved by means of the composition rule of $\bar{f}$.

The main contribution of this work is to introduce the fuzzy extension of a classical function as a useful alternative to the set of fuzzy rules or knowledge base for the modelling of inherently imprecise relations.

The essential difference between a set of fuzzy rules and a fuzzy extension is that while the former expresses the correspondence by a set of pairs of fuzzy sets $(\bar{x}_i, \bar{y}_i)$, the fuzzy extension does it by a unique fuzzy set in $\bar{f}$ and so, the image $\bar{y}$ of any fuzzy set $\bar{x}$ through a fuzzy extension is directly obtained by performing the fuzzy composition of $\bar{x}$ and $\bar{f}$.

As mentioned above, the inherently imprecise relations between variables are very common in a large number of knowledge areas, environmental assessments being one of the fields where such relations occur more frequently. Despite their imprecision, very often these relations are not described by means of sets of fuzzy rules, but by classical functions $f(x)$. In such cases, $f(x)$ represents a simplification of the real correspondence, since it does not adequately describe the inherent imprecision of the relation. When this is the case, the fuzzy extension $\bar{f}$ of a classical function $f(x)$ enables us to set an accurate description of the inherently imprecise relation, $\bar{f}$ being obtained from the "extension" of a classical function $f(x)$ by means of fuzzy set theory.

The further development of the arithmetic of fuzzy extensions presented in this paper and their applications is open to future research, which will expand the possibilities of the application of fuzzy logic and fuzzy sets theory to the modelling of inherently inaccurate relationships.

The work is organised as follows: first the concept of fuzzy extension and the procedure to specify and model its membership function $\mu_{\bar{f}}(x,y)$ are presented in Section 2. Subsequently, a procedure for determining the image $\bar{y}$ of $\bar{x}$ through a fuzzy extension $\bar{f}$ is proposed in Section 3, where the fuzzy composition rule is defined in order to calculate the image of a fuzzy singleton (FS) and a triangular fuzzy number (TFN). Next, the basic arithmetic operations are defined in Section 4: the addition, subtraction, multiplication and division. Next, a comparative study of the concepts of fuzzy extension and knowledge base is presented in Section 5. Later, in order to illustrate the application of the concept of "*fuzzy extension of a classical function*", the modelling of the correspondence between an environmental indicator (I) and its contribution $\overline{Q_I}$ to a quality index is presented in Section 6, which is obtained from the fuzzy extension of the classical transformation function $Q_I = f_t\,(I)$. Finally, some conclusions are drawn in Section 7.

2. The concept of the fuzzy extension $\bar{f}$ of a classical function *f*

The fuzzy extension $\bar{f}$ of a classical continuous function f is a particular kind of fuzzy relation [23]. $\bar{f}$ describes the correspondence between two variables x and y, so that the univocal image of each value of x is a closed interval of y values. The extent to which each value of the y interval corresponds to the value of x varies between 0 and 1, $f(x)$ being the value of y which corresponds to x with the highest extent of correspondence, i.e. 1.

Two functions are defined in order to set the limits of the y interval. They are called upper and lower fuzziness limits of the fuzzy extension $\bar{f}$, f_u and f_l, respectively. These functions determine the interval of y values whose extent of correspondence to x is nonzero, being zero $\forall y : y \leqslant f_l(x)$ and $\forall y : y \geqslant f_u(x)$.

According to this, let's consider the real functions $f(x)$, $f_l(x)$ and $f_u(x)$, the domain and codomain of which are $D = [x_{\min}, x_{\max}]$ and $B = [y_{\min}, y_{\max}]$, respectively. These functions are continuous in D and they fulfill the condition that:

$$\forall x \in D\, f_l(x) \leqslant f(x) \leqslant f_u(x). \tag{1}$$