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An order optimal regularization method for the Cauchy problem of a Laplace equation in an annulus domain

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ABSTRACT

The detection of defects due to corrosion in oil pipeline industry is very important. Detecting corrosion by electrical field can be modeled as a Cauchy problem for a Laplace equation in an annulus domain, which is well known to be severely ill-posed. In this paper, a modified Tikhonov regularization method is proposed. And a Hölder-type error estimate is achieved, which is order optimal according to the general regularization theory. Moreover, a Fast Fourier Transform (FFT) is used in the numerical implementation. Finally, numerical results show that the regularization method and FFT work well.

1. Introduction

In order to transport oil in a safe and reliable way, corrosion prevention is a quite important task. According to the *Association of Oil pipe Lines (AOPL)*, the most recent *Crack Detection Tools* include ultrasonic crack detection, magnetic flux leakage, and elastic wave tool. Varieties of detects in pipeline can be classified into three categories: defects due to corrosion, defects generated by mechanical damage, and cracks created by stresses in the pipe wall, see [1] for details.

When considering the defects due to corrosion, the nondestructive determination for the corrosion that occurs on the interior surface of the pipeline by using the electric field can be modeled as an inverse problem, since the only accessible data are the electrostatic measurements on the exterior surface of the pipeline.

Assuming the pipe is a metallic body with constant conductivity, Inglese [2] modeled the problem of determining quantitative information about corrosion by a Laplace equation for the electrostatic potential as follows:

$$\Delta w(\vec{x}) = 0, \quad \text{in } \Omega, \tag{1}$$

with exterior data

$$w(\vec{x}) = g(\vec{x}), \; w_n(\vec{x}) = h(\vec{x}), \quad \text{on } \Gamma_{out}, \tag{2}$$

where $w(\vec{x})$ is the electric potential, $w_n(\vec{x})$ is the outer normal derivative of the electric potential, $\Gamma_{out} = \{r = r_2\}$ is the exterior boundary, $\Gamma_{in} = \{r = r_1\}$ is the interior boundary, and $\Omega = \{r_1 \leqslant r \leqslant r_2\}$ is the cross sectional region of the pipe, which is assumed to be an annulus domain. An illustration of the pipeline cross sectional region and its boundaries is shown in Fig. 1.

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In addition, $g(\vec{x})$ is the electric potential on the exterior boundary, while $h(\vec{x})$ is the normal current flux on the exterior boundary. Assuming that the entire exterior boundary Γ_{out} is accessible, $g(\vec{x})$ and $h(\vec{x})$ are known for all $\vec{x}$ on Γ_{out}. Here we use the same assumptions as in [1–3].

On the interior boundary Γ_{in}, Buttazzo and Kohn [4] observed that mixed boundary conditions are needed as the thickness of the coating goes to zero. Based on the Faraday's law, corrosion or mass loss is proportional to the normal current flux. Thus, one can model the condition on the interior boundary as follows:

$$w_n(\vec{x}) + \gamma(\vec{x})w(\vec{x}) = 0, \quad \text{on } \Gamma_{in}, \tag{3}$$

where $\gamma(\vec{x})$ is the coefficient of energy exchange and $\gamma(\vec{x}) \geqslant 0$ indicates corrosion damage. Once the solution of Cauchy problem (1) and (2) is obtained, one can easily use (3) to find $\gamma(\vec{x})$ as follows:

$$\gamma = -\frac{w_n}{w}, \quad \text{on } \Gamma_{in}. \tag{4}$$

Therefore, in this paper, we only discuss the solution of the Cauchy problem (1) and (2).

The Cauchy problem of an elliptic equation is well known to be ill-posed in the sense of Hadamard. Under an additional condition, a continuous dependence of the solution on the Cauchy data can be obtained. This is called conditional stability. For example, some results on the conditional stability for the Cauchy problem of the Laplace equation can be found in [5–7]. Moreover, there are a wide range of literature on Cauchy problems for the Helmholtz equation and the modified Helmholtz equation [8–16].

Since there will always be some measure errors in the given Cauchy data, it is impossible to solve the problem directly due to the ill-posdeness of the problem. Some regularization methods have to be used in order to get stable convergent solutions. There are lots of literature on both the theoretical and methodological developments in this field. For theoretical aspects, the readers can refer to [6,17–20]. For computational aspects, the readers can refer to [5,21–28]. For the Cauchy problem of the Laplace equation in an annulus domain, two good references are [1,3], which use an energy regularization method and the method of fundamental solutions (MFS) to obtain regularized solutions from the noisy Cauchy data, respectively. However, the error estimates are not order optimal. In this paper, we propose a modified Tikhonov regularization method for the same problem. And a Hölder-type error estimate is achieved, which is order optimal according to the general regularization theory [29]. Furthermore, a Fast Fourier Transform (FFT) is used in the numerical implementation. Numerical results show that the regularization method and FFT work well. The aim of this paper is to present an order optimal regularization method.

The paper is organized as follows. In Section 2, a modified Tikhonov regularization method is introduced. In Section 3, some error estimates are provided under a priori assumption for the exact solution. Section 4 shows the numerical results, while conclusions and discussions are given in the final section.

2. Modified Tikhonov regularization method

Since Ω is an annulus domain, the Cauchy problem (1) and (2) can be rewritten in polar coordinate as follows:

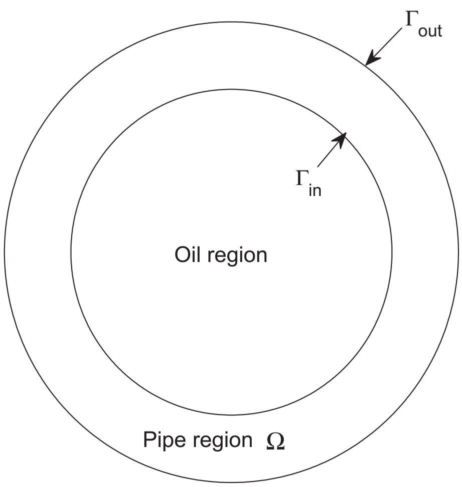

Fig. 1. An illustration of the pipe cross sectional region and its boundaries.