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Helical flow arising from the yielded annular flow of a Bingham fluid

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ABSTRACT

The Bingham fluid model was developed to represent viscoplastic materials that change from rigid bodies at low stress to viscous fluids at high stress – a process termed yielding. Such a fluid model is used in the modeling of slurries, which occur frequently in food processing and other engineering applications.

We consider the flow of a Bingham fluid between infinitely long coaxial cylinders, when the inner cylinder rotates. This is of relevance to a number of applications, including rheometry. We assume a yielded flow, comprising an inner fluid zone and an outer solid zone; and apply a perturbation procedure to analyze the changes in the characteristics of this during a transition from pure annular fluid motion (no axial flow) to a helical fluid motion, where a small axial fluid flow rate is imposed.

This analysis gives explicit expressions for the changed fluid velocity field as well as movement in the location of the solid–fluid boundary.

1. Introduction

Helical flow of an incompressible viscous fluid in the gap between infinitely long coaxial cylinders occurs when an axial fluid velocity is superimposed on a transverse rotational flow generated by the rotation of one of the bounding cylinders. Such flows are of interest in applications, particularly in modeling the action of a cup and bob rheometer modified to allow axial flow, in order to carry out measurements on slurries and other settling mixtures [1].

For Newtonian fluids, such helical flow is well understood, and the equations for the fluid field are readily integrated to give an exact solution – see Langlois [2]. However, for non-Newtonian fluids, with nonlinear constitutive equations linking stress and rate of shearing, the task of obtaining the velocity profile and associated fluid properties is considerably more difficult and numerical solution techniques must be resorted to. Although a general form of solution of this problem was provided by Coleman and Noll [3], their analysis replaced a nonlinear boundary value problem with a nonlinear algebraic one, leaving the details of the solution of either unresolved. When the fluid undergoing such helical flow displays a *yield stress*; that is, it changes from a rigid body at low stress to a viscous fluid at high stress, a solid–fluid boundary (the *yield boundary*) can arise in the intercylindrical gap, with a solid zone adjacent to the stationary cylinder.

In the absence of axial flow, that is, when the flow is annular, the location of this boundary is determined by the cylinders' relative dimensions, the rotational speed of the rotating cylinder and the parameters occurring in the fluid's constitutive

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equation. Superposition of an axial flow, changing this annular flow to helical flow will cause this boundary to move in a manner governed by the axial flow rate.

In what follows we employ a perturbation method, based on a scaled form of the axial flow rate as perturbation parameter, to analyze the variations in the location of the yield boundary and of the fluid flow field resulting from a transition from a yielded annular flow to a helical flow. We consider a Bingham fluid, the simplest form of yield stress fluid model, which has been used to represent the flow of a wide range of fluids, ranging from tomato sauce to wet concrete. Solution of the annular flow problem involving a yield boundary for this fluid model is long solved – see Wilkinson [4]. Our calculations will result in explicit easily applied approximate expressions for the yield boundary location and the fluid velocity field in the flow region, valid for small values of the axial flow rate. These calculations apply and extend the methods and results of Chiera [5] and Chiera et al. [1] who considered a helical flow of a Bingham fluid when all the fluid in the intercylindrical gap is yielded.

2. Governing equations

We consider steady helical flow of an incompressible viscous fluid in the infinitely long annular region described in cylindrical polars (r,θ,z) by $R_i \leqslant r \leqslant R_o,\ 0 \leqslant \theta \leqslant 2\pi,\ -\infty \leqslant z \leqslant \infty$. The inner cylindrical surface $r = R_i$ is given a constant angular velocity $\Omega > 0$, the outer cylinder $r = R_o$ is held stationary, and a given axial flow rate $Q > 0$ is imposed. We also assume that there is a freeze radius $\bar{R}$, such that the fluid in the region $r > \bar{R}$ is stationary. Note when the axial flow increases sufficiently the freeze radius will equal the radius of the outer wall. We are assuming that Q is small enough that this does not occur.

The equations of motion for such a flow are well-documented in the research literature [5]. With the z-axis taken as vertically down and assuming all quantities exhibit symmetry about this axis, the velocity field which satisfies the equation of conservation of mass take the form

$$(u_r, u_\theta, u_z) = (0, rW(r), V(r))$$

for appropriate functions $V(r),\ W(r)$. One integration of the momentum equations yields differential equations for the functions $V(r)$ and $W(r)$

$$HV'(r) = \alpha r + \frac{\beta}{r}, \quad HW'(r) = -\frac{M}{2\pi r^3}. \tag{1}$$

Here H is the fluid viscosity, $M > 0$ may be interpreted as the moment per unit length exerted on the inner cylinder $r = R_i$, and α and β are constants of integration, to be determined by the application of the boundary conditions at the cylinder walls. In our calculations, we view M as a known quantity, determined in practice by experimental measurements – for example, in the cup and bob rheometer by readings of a dynamometer attached to the inner cylinder.

In what follows the equations of motion will only apply to the fluid in the yielded region, given by $R_i \leqslant r \leqslant \bar{R}$.

Eliminating H between the equations given in (1) yields the relationship

$$V'(r) = -2\pi(\alpha r^2 + \beta)r^2 W'(r)/M. \tag{2}$$

Further, $V(r)$ is related to the volume flow rate Q by

$$Q = 2\pi \int_{R_i}^{\bar{R}} rV(r)dr. \tag{3}$$

(Q is regarded as known, so that (3) is a constraint on $V(r)$).

The local rate of shearing, K, becomes $K = \sqrt{[(rW')^2 + V'^2]/2}$, in terms of the gradients of V and W. The second of (1) implies that $W'(r) < 0$ in all $R_i < r < \bar{R}$; so that $|W'| = -W'$. This, with (2) converts K to

$$K = -rW'\phi(r,\alpha,\beta)/\sqrt{2}, \quad \text{where} \quad \phi(r,\alpha,\beta) = \sqrt{1 + (2\pi/M)^2 r^2 (\alpha r^2 + \beta)^2}. \tag{4}$$

For generalized Newtonian fluids, the viscosity, H is a function of the local rate of shearing, K; i.e., $H = H(K)$. For the Bingham fluid, this is

$$H(K) = H_0 + \frac{T_0}{K} \tag{5}$$

in the yielded region, where H_0 and T_0 are positive constants.

Note that putting $T_0 = 0$ in (5) gives $H(K) = H_0$, and the Bingham fluid reduces to a Newtonian one of constant viscosity H_0.

The non-slip conditions on the inner cylinder and the cylindrical freeze boundary yield the boundary conditions

$$V(R_i) = V(\bar{R}) = 0, \quad W(R_i) = \Omega, \quad W(\bar{R}) = 0. \tag{6}$$

Further, at the freeze radius $r = \bar{R}$, the rate of shearing is zero, so that $K = 0$; or, equivalently,

$$V'(\bar{R}) = W'(\bar{R}) = 0. \tag{7}$$