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A lifetime model with increasing failure rate

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ABSTRACT

This paper deals with a new two-parameter lifetime distribution with increasing failure rate. This distribution is constructed as a distribution of a random sum of independent exponential random variables when the sample size has a zero truncated binomial distribution. Various statistical properties of the distribution are derived. We estimate the parameters by maximum likelihood and obtain the Fisher information matrix. Simulation studies show the performance of the estimators. Also, estimation of the parameters is considered in the presence of censoring. A real data set is analyzed for illustrative purposes and it is noted that the distribution is a good competitor to the gamma, Weibull, exponentiated exponential, weighted exponential and Poisson-exponential distributions for this data set.

1. Introduction

Increasing failure rate (IFR) distributions are of interest in many real life systems; see for instance Koutras [1] and Ross et al. [2]. The most popular distributions that have IFR are the gamma and Weibull distributions (these distributions also exhibit decreasing and constant failure rates). Gupta [3] and Gupta and Kundu [4] introduced the exponentiated exponential distribution and the weighted exponential distribution as alternatives to the gamma and Weibull distributions, respectively. Recently, Cancho et al. [5] introduced a new lifetime distribution with increasing failure rate, named as the Poisson-exponential (PE) distribution.

The aim of this paper is to propose a new two-parameter lifetime distribution with IFR property, named as the binomial-exponential 2 (BE2) distribution, which is constructed as a distribution of a random sum of independent exponential random variables when the sample size has a zero truncated binomial distribution. As we will see later, this distribution can be used as an alternative to the Weibull, gamma, exponentiated exponential, weighted exponential and PE distributions.

Let N denote a zero-truncated binomial random variable with the probability mass function specified by

$$P(N=k)=\frac{\binom{n}{k}\theta^k(1-\theta)^{n-k}}{1-(1-\theta)^n} \tag{1}$$

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for $k = 1, 2, \ldots n$ and $0 < \theta < 1$. The zero-truncated binomial distribution is actually a popular model though it has been neglected to some extent in the statistical and actuarial literatures. For example, there is no mention of the zero-truncated binomial distribution in Klugman et al. [6]. Some recent applications of the zero-truncated binomial distribution have been to model: the choice set size for unemployed (Magnussen [7]); anglers' consumer surplus values per trip (Oh et al. [8]); the number of quadrats containing a given species (Shen and He [9]); the number of times a particular animal is captured given that it is captured (Ashbridge and Goudie [10]); the number of secondary users that are sensing a given primary channel (Bkassiny et al. [11]); the number of plots containing a given species (Magnussen [12]); the number of days a user uses the search engine for search considering that the search log only includes those users who initiated at least one search (Wang et al. [13]). See also Mao and Colwell [14].

Now let $\{W_i\}$ be a sequence of independent and identically distributed (i.i.d.) exponential random variables independent of N. Consider the random sum $X = \sum_{i=1}^{N} W_i$. The distribution of X arises naturally in each of the above cited applications. For example, if the lifetimes of quadrats (Shen and He [9]) are assumed to be i.i.d. exponential random variables then the random variable X will represent the total lifetime. Since the exponential distribution is the simplest lifetime model, the simplest possible distribution for X could be obtained by taking W_is as exponential random variables.

The paper is organized as follows. In Section 2, we introduce the BE2 distribution and study various statistical properties of it. The properties studied include shapes of the probability density function (p.d.f.) (Section 2.1), shapes of the hazard rate function, reliability properties (Section 2.2), raw moments (Section 2.3), moments of (reversed) residual life (Section 2.4), stress-strength parameter (Section 2.5), distributions of sums and ratios (Section 2.6), order statistics, their moments and the asymptotic distribution of the extreme values (Section 2.7), mean deviations about the mean and the median (Section 2.8), Bonferroni curve, Lorenz curve, Gini's index (Section 2.9), and entropy (Section 2.10). Estimation of the parameters by maximum likelihood method is investigated (Section 3.1) with simulation studies (Section 3.2) to investigate the performance of the estimators. Also, estimation of the parameters is considered in the presence of censoring (Section 3.3). An application to a real data set is presented in Section 4. Section 5 ends the paper with some concluding remarks.

2. The proposed model and its properties

The proposed model can be derived as follows. Let N denote a zero-truncated binomial random variable given by Eq. (1). Given N, let $X = \sum_{i=1}^{N} W_i$, where $W_1, \ldots, W_N$ are i.i.d. random variables independent of N, having the exponential distribution with the parameter λ. Then, the conditional distribution of $X|N = n$ is

$$f_{X|N=n}(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, \quad x > 0.$$

Hence, the marginal p.d.f. of X is

$$f_X(x) = \sum_{k=1}^{n} f_{X|N=k}(x) P(N = k) = \sum_{k=1}^{n} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \frac{\binom{n}{k} \theta^k (1-\theta)^{n-k}}{1 - (1-\theta)^n}.$$

This p.d.f. is very complicated, but for the case $n = 2$ we obtain a simple generalization of the exponential p.d.f.

$$h(x; \lambda, \theta) = \left(1 + \frac{(\lambda x - 1)\theta}{2 - \theta}\right) \lambda e^{-\lambda x} \tag{2}$$

for $x > 0, 0 \leqslant \theta \leqslant 1$ and $\lambda > 0$. The corresponding cumulative distribution function (c.d.f.) is

$$H(\lambda, \theta; x) = 1 - \left(1 + \frac{\lambda \theta x}{2 - \theta}\right) e^{-\lambda x}.$$

It is easy to show that the survival function, $\overline{H}(\cdot) = 1 - H(\cdot)$, is increasing in $0 \leqslant \theta \leqslant 1$ (see Fig. 4), so we have the following bounds

$$e^{-\lambda x} \leqslant \overline{H}(x; \lambda, \theta) \leqslant (1 + \lambda x) e^{-\lambda x}.$$

Note that the ordinary exponential distribution is the particular case of (2) for $\theta = 0$. The particular case for $\theta = 1$ is the gamma distribution with shape parameter 2 and scale parameter λ. If X is a random variable with p.d.f. (2) then we say that X has the BE2 distribution. A BE2 random variable with parameters λ and θ shall be denoted by BE2(λ, θ).

The BE2 distribution given by (2) is new. Klugman et al. [6] give a catalogue of distributions arising from random sums i.i.d. random variables. Lai and Xie [15] give a catalogue of distributions extending the standard exponential, gamma and Weibull distributions. The BE2 distribution is not contained in these catalogues. We have also checked many papers in the statistical, actuarial, insurance, and reliability literatures to confirm originality of the BE2 distribution.

The importance of the BE2 distribution defined by (2) lies in its ability to model lifetime data with increasing failure rate. Also, we shall see later that the parameter λ can be interpreted as an upper bound on the failure rate function, an important characteristic for lifetime models. Not many lifetime distributions (for example, the Weibull distribution) have their parameters directly interpretable in terms of their hazard rate functions. Another attractive feature of the BE2 distribution is that it