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A non-interior continuation algorithm for solving the convex feasibility problem ☆

Nan Lu *, Feng Ma, Sanyang Liu

School of Mathematics and Statistics, Xidian University, XiAn 710071, PR China

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ABSTRACT

In this paper, a specific class of convex feasibility problems are considered and a non-interior continuation algorithm based on a smoothing function to solve this class of problems is introduced. The proposed algorithm solves at most one system of linear equations at each iteration. Under some weak assumptions, we show that the algorithm is globally linearly and locally quadratically convergent. Preliminary numerical results are also reported, which verify the favorable theoretical properties of the proposed algorithm.

1. Introduction

The convex feasibility problem (*CFP*, for short) is the problem of finding a point laying in the intersection of a finite family of closed convex subsets $\{C_i\}_{i=1}^N$, where N is a positive integer. Usually, these convex sets are given as sublevel sets of convex functions,

$$\{C_i\}_{i=1}^N = \{x \in R^n : f_i(x) \le 0\}_{i=1}^N, \tag{1.1}$$

where $f_i : R^n \to R$ is convex function for every $i = 1, 2, .., N$. The *CFP* is fundamental in many areas of application such as image recovery[1], radiation therapy treatment planning [2,3] and demosaicking [4]. It is studied in many different contexts, see, for instance, [5–7] and references therein.

Most of the major proposed methods for solving *CFP* (1.1) can be grouped into the following classes, successive projection methods [8,9], simultaneous projection methods [10–12], string-averaging projection methods [13,14], block-iterative methods [13,15–18] and subgradient projection methods [11,19,20]. For more information on these terms and a survey on the algorithms for solving convex feasibility problems refer to [21,22].

Non-interior continuation algorithms have been proposed for solving various optimization problems successfully, for example, [23–29]. Recently, there has been increasing interest in solving optimization problems by using non-interior continuation algorithms. Thus, *a natural question is whether or not non-interior continuation algorithms can be used to solve CFP* (1.1)? *And how to apply non-interior continuation algorithms to solve the CFP* (1.1)? This paper will show some results on this topic.

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* Corresponding author. Tel.: +86 02988202860.
E-mail addresses: mathlunan@gmail.com (N. Lu), mafengforever@sina.com (F. Ma), liusanyang@126.com (S. Liu).

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For *CFP* (1.1) with $N = n$, we propose a non-interior continuation algorithm to solve such a problem in this paper, i.e., to find a point which satisfies

$$\{C_i\}_{i=1}^n = \{x \in R^n : f_i(x) \le 0\}_{i=1}^n, \tag{1.2}$$

where $f_i : R^n \to R$ is a convex function for any $i \in \{1, 2, \dots, n\}$.

For any $x \in R^n$, define

$$x_+ = (\max\{0, x_1\}, \dots, \max\{0, x_n\})^T, \quad |x| = (|x_1|, \dots, |x_n|)^T \quad \text{and} \quad f(x) = (f_1(x), \dots, f_n(x))^T,$$

then

$$x \in \{C_i\}_{i=1}^n \Longleftrightarrow f(x) \le 0 \Longleftrightarrow f(x)_+ = 0,$$

so *CFP* (1.2) is equivalent to the following system of equations:

$$f(x)_+ = 0. \tag{1.3}$$

Since the the function involving in (1.3) is non-smooth, the classical Newton methods can not be directly applied to solve (1.3). In this paper, we introduce the following smoothing function,

$$\phi(\mu, a) = \frac{a + \sqrt{a^2 + \mu^2}}{2}, \quad \mu > 0. \tag{1.4}$$

Proposition 1.1. *For any given $\mu > 0$, we have*

(i) *$\phi(\mu, \cdot)$ is continuously differentiable everywhere with $\phi'_a(\mu, a) = \frac{1}{2}\left(1 + \frac{a}{\sqrt{\mu^2 + a^2}}\right)$, $\forall a \in R$.*
(ii) *When $\mu = 0$, $\phi(\mu, a) = a_+$.*

Define

$$\Phi(\mu, y) = \begin{bmatrix} \phi(\mu, y_1) \\ \vdots \\ \phi(\mu, y_n) \end{bmatrix}. \tag{1.5}$$

then by Proposition 2.1 (ii), we have

$$\Phi(\mu, y) = 0, \quad \mu = 0 \quad \text{and} \quad y = f(x) \Longleftrightarrow y = f(x), y_+ = 0.$$

This, together with Proposition 2.1 (i), indicates that, to solve *CFP* (1.2), one can apply Newton-type methods to solve $\Phi(\mu, y) = 0,\ y = f(x)$ and make $\mu \downarrow 0$.

This paper is organized as follows. In the next section, we propose a non-interior continuous algorithm for solving *CFP* (1.2). In Section 3, we show that the algorithm is globally convergent, and investigate the global linear and local quadratic convergence of the algorithm. The preliminary numerical results are reported in Section 4, and in Section 5, we give a conclusion.

In the following, $K := \{1, 2, \dots\}$ denotes the iteration index set, I $n \times n$ identity matrix, and R_{++} the set of positive reals. For any vectors $x, y \in R^n$, we write $z = (x^T, y^T)^T$ as $z = (x, y)$ for simplicity. $\nabla_z H(\mu, z)$ means the gradient of $H(\mu, z)$ with respect to $z \in R^n \times R^n$ and $\nabla f(x)$ the gradient of $f(x)$ with respect to $x \in R^n$. For any $u_k, v_k \in R$ with $k \in K$, $u_k = o(v_k)$ means $\lim_{k\to\infty} u_k/v_k = 0$ and $u_k = O(v_k)$ means $\limsup_{k\to\infty} u_k/v_k < \infty$.

2. Algorithm description

Let $\phi(\mu, a)$ and $\Phi(\mu, y)$ be defined by (1.4) and (1.5), define

$$H(\mu, z) = \begin{bmatrix} y - f(x) \\ \Phi(\mu, y) \end{bmatrix}. \tag{2.1}$$

It is easy to see that $\mu = 0$ and $H(\mu, z) = 0$ if and only if x solves *CFP* (1.2).

Algorithm 2.1 (*A non-interior continuation algorithm*).

Step 0 *Given $\sigma, \alpha, \gamma \in (0, 1)$, $\mu_0 > 0$, and $x^0, y^0 \in R^n$. Set $z^0 := (x^0, y^0)$. Choose β such that $\beta \geqslant \sqrt{n}/2$ and $\|H(\mu_0, z^0)\| \leqslant \beta\mu_0$. For any $k \ge 0$.*
Step 1 *If $\|H(0, z^k)\| = 0$, stop; otherwise, go to Step 2.*