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Homotopy perturbation method for two dimensional time-fractional wave equation ☆

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ABSTRACT

The aim of this paper is to present an efficient and reliable treatment of the homotopy perturbation method (HPM) for two dimensional time-fractional wave equation (TFWE) with the boundary conditions. The fractional derivative is described in the Caputo sense. The initial approximation can be determined by imposing the boundary conditions. The method provides approximate solutions in the form of convergent series with easily computable components. The obtained results shown that the technique introduced here is efficient and easy to implement.

1. Introduction

In the last few decades many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. In recent years, fractional differential equations (FDEs) have been caught much attention. Many physical problems are governed by fractional differential equations (FDEs), and finding the solution of these equations has been the subject of many investigators. The main reason consists in the fact that the theory of derivatives of fractional (noninteger) stimulates considerable interest in the areas of mathematics, physics, engineering and other sciences. Several analytical and numerical methods have been proposed to solve fractional differential equations of physical interest, such as finite difference method (FDM) [1], adomian decomposition method (ADM) [2,3], Laplace transform method (LTM) [4–6], variational iteration method (VIM) [7,8] and homotopy perturbation method (HPM) [9,10]. Also there are some other classical solution techniques.

The most commonly used one is He's homotopy perturbation method. The HPM is the new approach for finding the approximate analytical solution of linear and nonlinear problems. The method was first proposed by He [11,12] and was successfully applied to solve the traditional differential equations by He [12–18]. Some new development of the HPM by authors, please see [19–21]. The HPM is also applied to the fractional differential equations, such as partial differential equations [22], fractional diffusion and wave equations [23], fractional KdV–Burgers equation[24], space–time fractional

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advection–dispersion equation [25], time-fractional diffusion equation with a moving boundary condition [26], fractional diffusion equation with absorbent term and external force [27], fractional convection–diffusion equation with nonlinear source term [28], nonlinear fractional Kolmogorov–Petrovskii–Piskunov equations [29] and fractional Fornberg–Whitham equation [30]. It is worth mentioning that the HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors.

In the present paper we will use the HPM to construct an approximate solution to the time-fractional wave equation (TFWE). The TFWE, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical wave equation by replacing the second time derivative by a fractional derivative of order α $(1 < \alpha \leqslant 2)$. It has been studied by many researchers by Wyss [31], Schneider and Wyss [32], Fujita [33], El-Sayed [34], Gorenflo and Mainardi [35] and Hanyga [36]. Gorenflo et al. [37] used the similarity method and the Laplace transform method to obtain the scale-invariant solution of the time-fractional diffusion-wave equation in terms of the Wright function. Agrawal [38] presented a general solution for a fractional diffusion wave equation defined on a bounded space. The space–time fractional-wave equation has also treated by Mainardi et al. [39] as a Cauchy problem, and its fundamental solution was investigated in terms of Green's function. Odibat and Momani [40] considered the boundary value problems of time-fractional wave equation by ADM.

We will study the following two dimensional time-fractional wave equation:

$$\begin{cases} D_t^{\alpha}u(x,y,t) = \lambda L_{xy}u(x,y,t) + f(x,y,t), \quad 0 < x,y,\ 0 < t < b, \\ u(x,y,0) = g(x,y),\ u(x,y,b) = h(x,y), \end{cases} \tag{1}$$

where $1 < \alpha \leqslant 2,\ \lambda$ and b are positive constants, D_t^{α} denotes the Caputo fractional derivative in time and $f(x,y,t),\ g(x,y)$ and $h(x,y)$ are given functions. L_{xy} is the linear differential operator

$$L_{xy}u(x,y,t) = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} = u_{xx} + u_{yy}. \tag{2}$$

The motivation of this paper is to extend the application of the He's homotopy perturbation method [11,12] to solve two dimensional boundary value problems. To the authors knowledge, this paper represents the first application of He's homotopy perturbation method to solve two dimensional time-fractional wave equation. Several examples are given to verify the reliability and efficiency of the homotopy perturbation method.

The rest of this paper is organized as follows: In Section 2, some basic definitions and properties of fractional calculus theory are given. In Section 3, the basic idea of the HPM is given. In Section 4, we obtain the solution of time-fractional wave equations with boundary conditions.

2. Preliminaries

There are many kinds definition of fractional derivative and integral, such as the Riemann–Liouville (R–L) fractional derivative and integral, Caputo fractional derivative and integral, etc. In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this article, about these definitions and properties one can refer to [4–6].

Definition 2.1. A real function $u(t),\ t > 0$ is said to be in space C_{θ} $(\theta \in R)$ if there exists a real number $p > \theta$, such that $u(t) = t^p u_1(t)$, where $u_1(t) \in C(0,\infty)$, and it is said to be in the space C_{θ}^n if and only if $u^n \in C_{\theta},\ n \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $u(t) \in C_{\theta},\ \theta \geqslant -1$, is defined as

$$\begin{cases} J^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)d\tau,\ \alpha > 0,\ \tau > 0, \\ J^0 u(t) = u(t). \end{cases} \tag{3}$$

For $u(t) \in C_{\theta},\ \theta \geqslant -1,\ \alpha,\beta \geqslant 0$ and $\gamma \geqslant -1$, some properties of the operator J^{α}, which are needed here, are as follows:

$$(i)\ J^{\alpha}J^{\beta}u(t) = J^{\alpha+\beta}u(t); \quad (ii)\ J^{\alpha}J^{\beta}u(t) = J^{\beta}J^{\alpha}u(t); \quad (iii)\ J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

Definition 2.3. The fractional derivative in the Caputo sense of $u(t) \in C_{-1}^m,\ m \in N,\ t > 0$ is defined as

$$D_t^{\alpha}u(t) = \begin{cases} J^{\alpha-m} \frac{d^m}{dt^m} u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} u^m(\tau)d\tau, \quad m-1 < \alpha < m, \\ \frac{d^m}{dt^m} u(t), \quad \alpha = m. \end{cases} \tag{4}$$