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A new algorithm for dual-rate systems frequency response computation in discrete control systems

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This paper addresses an easy computation of the multiple components of the response to a sinusoidal input of a dual-rate linear time-invariant discrete system from the Bode diagram of LTI systems arising from a lifted representation. Based on those results, a generalized Bode diagram is suggested. Some new conclusions derived from this conceptual interpretation are introduced. This diagram provides a better insight in the frequency-response issues in multivariable control than the standard singular value decomposition of the lifted model. As an application, the output ripple suppression in a multirate control scheme is presented.

1. Introduction

The consideration of multirate systems (MRS) in fields like signal processing and communications are well established many years ago. Digital control is also an environment where multirate systems are used either to overcome practical difficulties or to achieve unattainable results by single rate control [1–4]. In fact Dual-rate systems in systems and control have long ago been of interest to engineers [5]: low-latency measurements, limited-speed actuators in control loops, fast sensing in order to better filter measurement noise, network load [6,7], computational resources [8], zero-assignment [9], *etc.* are issues concerning possible applications advantages of dual-rate systems.

Although there are some techniques described in [10–13], the usual procedure to handle a MRS that is a time-variable system is to consider the technique known as lifting (in control area) [14–17] or blocking (in signal processing) [18]. With this procedure the system is transformed into a LTI system once the system description is enlarged over a "metaperiod". Using this technique, even from a single-input single-output (SISO) system an artificial multi-input and multi-output (MIMO) system is obtained.

One of the classical problems in multirate control is derived from the consideration that such MIMO lifted system can be controlled as any other multivariable MIMO one. However, there are not different input and output variables, but just one input and one output, "lifted" at different input and output sampling times in a periodically-repeating metaperiod. In automatic control field this procedure was originally denoted as Vectorial Switch Decomposition by Kranc [19]. Different authors proved that lifting and blocking were actually the exact same operation providing an unified analysis with that of periodic systems [20,21]. Together to on-line identification of this kind of systems [22–25], one of the recurrent topics that has been

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explored in this environment is the frequency response and its use in control [26,27]. Indeed, frequency-response issues are still a current matter of interest in control [28,29] and estimation problems [25].

From the above-presented considerations, it is common to study the frequency response of dual-rate plants by using singular value decomposition (SVD) of the lifted MIMO system. The conclusions are correct, but the results are partial due to the fact that the frequency response is obtained up to a metaperiod and lifting is disregarded. Hence, some relevant aspects for control cannot be obtained, as the single-rate lifted SVD leaves out some specific properties of the dual-rate setup.

An alternative approach is the lifting in "frequency" instead of the lifting in "time" that the above literature proposed. This is the so-called AC (alias component) matrix [30]. It is difficult and laborious to do this operation. Actually it is a hard task and for this reason it motivated contributions like [31] where a coprime periods were supposed in the dual-rate scheme.

Computation of the frequency response of non-conventionally-sampled systems was addressed in, for instance, [32,33]. In [33], the goal was to detect ripples in the controlled variables as well as to generalize engineering-related design criteria; the dual-rate frequency response to a sinusoidal input was be evaluated by checking the "conventional" Bode diagram of a particular discrete transfer function. However, the results only applied to the case when input sampling period T_u was an integer multiple or divisor of the output one T_y. This work will introduce a "generalized Bode diagram" which lifts such restriction, and includes several harmonic frequency components.

Preliminary work by the authors in [34] is extended in this contribution. A basic formula was introduced to easily obtain the multiple sinusoidal components of the dual-rate system's exact frequency response, under the assumption that the input and output periods were rationally related.

In the work here presented, the results are completed introducing a table of frequency components with an arrangement based on Bezout identity. Also, a generalized Bode diagram valid for coprime input/output periods is presented: the several harmonic components of a dual-rate response can be read as interleaved fragments of the frequency response of a particular single-rate system. Importantly, apart from a first academic example, the results are applied to feedback control systems, proposing a methodology to analyse the frequent output-ripple phenomena, allowing to overcome this anomalous performance which is not clearly detected from the SVD diagrams of the frequency response of the lifted system from previous literature.

The structure of the paper is as follows: next section recalls some preliminary material and founds definitions and notation; Section 3 presents the main result on frequencial components of the output of a dual-rate system; some numerical examples are explained in Section 4, and then in Section 5 we will show the advantages of this new tool in a dual-rate control scheme analysis example where ripple effect appears and it is exactly detected by this computation tool. A conclusion section closes the paper.

2. Preliminaries and notation

Let us consider the transfer function $G(z) = B(z)/A(z)$, in the Z-transformed input–output domain representing a single-rate discrete-time linear time-invariant (LTI) system. When the above system is excited by an input $u(k) = a^k$, where a is *not* a pole of the system, the response admits a particular solution $y(k) = G(a)a^k$, because there exists a polynomial $S(z)$ such that the Z-transform of the output can be expressed as:

$$y(z) = G(z)u(z) = \frac{B(z)z}{A(z)(z-a)} = \frac{zG(a)}{z-a} + \frac{S(z)}{A(z)}$$

In the same way, when the input is $u(k) = e^{j\omega T_u k}$, such particular solution is $G(e^{j\omega T_u})e^{j\omega T_u k}$, usually denoted as frequency response, and the stationary response to sinusoidal signals can be easily determined from it by taking real and imaginary parts. The scalar T_u may be interpreted as "input sampling period". Of course, the actual computation of that solution does only have sense for stable systems or systems stabilized in closed loop.

A dual-rate discrete LTI system is one in which the input and output sequences are assumed to have different sampling periods, T_u and T_y. If they are rationally related, it is possible to define the least common multiple $T_0 = lcm(T_u, T_y)$ usually known as "metaperiod" or "frame period" and there exist integers N_u, N_y such that $T_0 = T_u N_u = T_y N_y$ (indeed, then $T_u/T_y = N_y/N_u$ is a rational number). It is usual to define the "greatest common divisor sampling period" $T = gcd(T_u, T_y)$ as well, such that $T_0 = NT$ being $N = lcm(N_u, N_y)$; therefore $T_0 = NT = N_u T_u = N_y T_y$. With these conditions, the behavior of the dual-rate system may be characterised via a "lifted" transfer function matrix:

$$y_l(z^N) = G_{lifted}(z^N)u_l(z^N), \tag{1}$$

where the variable z^N stands for the LTI z-transform argument at sampling period T_0, y_l is a vector of length N_y, u_l is a vector of length N_u and G_{lifted} is a $N_y \times N_u$ transfer function matrix [15]. The lengths of the vectors are increased in the case of MIMO systems (multiplied by the number of outputs and inputs, respectively). For convenience, zero-based array element count will be used in the sequel. For instance, the original T_y-sequence $y(k)$ and its lifted one $y_l(k)$ (T_0 related) are built in such a way that the i-th element of $y_l(k)$, to be denoted as $y_{l,i}(k)$, is $y(k * N + i)$, with $i = 0,\ N_u \ldots, N_u(N_y - 1)$.

In order to recover original sequences from lifted results in the Z-transformed domain, an expand operator [35] may be used, sometimes padding with zeros the intermediate samples.