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Numerical approximation strategy for solutions and their derivatives for two-dimensional solids

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ABSTRACT

The paper presents the approximation strategy for solutions and derivatives of solutions of boundary value problems on the example of two-dimensional solids. The effectiveness of the proposed strategy lies in the fact that it gives possibility to calculate solutions and their derivatives continuously at all points of the boundary and the domain, irrespective of the method used to solve the boundary problem and regardless of the type of the problem. The strategy has been developed in order to: (1) improve the accuracy of solutions (displacement) and their derivatives (strains, stresses) in the vicinity of the boundary, where they are affected by errors, (2) make the ability to directly obtain the derivatives of solutions (strains, stresses) on the boundary, (3) avoid computing strongly singular integrals present in the integral identity used for obtaining solutions or their derivatives (e.g. stresses in plasticity problems). The different variants of the proposed strategy have been developed and their accuracy has been verified considering examples with analytical solutions.

1. Introduction

In many cases, the solution to the boundary value problem is not only the result directly derived from the method, but also its derivative. An example is elasticity [1–3], where we directly obtain displacements or tractions, and secondly strains or stresses. Most commonly used numerical methods for solving this type of boundary problems are the finite element method (FEM) [2,4,5], the boundary element method (BEM) [2,6,7] or so-called meshless methods [8–10]. FEM and meshless methods are characterized by the fact that the derivatives of solutions can be less accurate than the solutions. This problem has been fixed in BEM, because it provides equal accuracy of solutions and their derivatives. However, this method is not free of defects and drawbacks. The first concerns less accurate solutions obtained in the close vicinity of the boundary, which is particularly visible in calculating the derivatives of solutions. The problem is solvable, but requires additional modifications of the method [11,12], which are not elementary. The another disadvantage is the fact that obtaining derivatives on the boundary is very often a big problem, because of the necessity of computing integrals, which contain strongly singular integrands. As is known, the accuracy of these integrals has a particular impact on the accuracy of the final results. Such situation, among others, is when we want to determine stresses on the boundary in elasticity problems [2,6]. There are several approaches to eliminate this difficulty, including the use of special methods for calculating singular integrals or the transformation of the troublesome equation to the nonsingular version [13,7,14]. We can also determine stresses on the basis of already obtained solutions on the boundary in conjunction with equations of elasticity [6].

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Choosing the method for solving boundary value problems we should pay attention especially on the way of modeling the boundary and the domain. FEM and BEM methods are so-called element methods, and are characterized by the necessity of discretization into elements. In the case of FEM, there are finite elements into which the entire domain is divided. BEM reduces the dimension of the problem by one, because we divide only the boundary into boundary elements. This is a big advantage of the method, which however, disappears when solving problems such as: problems of elasticity with body forces, problems modeled by the Poisson differential equation, plasticity problems and many others. These types of problems are characterized by the necessity of computing additional integrals over the domain. It requires dividing the domain into subdomains called cells, which in practice are equivalent to finite elements. Mentioned discretization becomes particularly troublesome when dealing with more complex problems such as 3D issues, problems with the moving boundary, crack propagation or large deformations. More effective methods for solving mentioned above issues are meshless methods [8]. They are characterized by the approximation made on a group of nodes arranged arbitrarily, which do not constitute the mesh, because there is no relationship between them. But they are not free of drawbacks. Some of them are: the necessity of the division of the domain defined by nodes into subdomains in order to compute integrals, difficulties in posing boundary conditions or complexity comes from modifications aimed at enhancing their capabilities [15,16,8].

Therefore, there is still the need to search for and develop new methods or modify already existing. Such an analytical modification of the boundary integral equations (BIEs) are the parametric integral equation systems (PIESs), whose main advantage is the complete elimination of the discretization both the boundary and the domain. PIESs were elaborated by one of the authors of the paper [17] and are developed through applications to solving the various kinds of boundary problems, e.g. 2D elasticity [18] or acoustics [19] and also 3D problems [20]. PIESs in these applications have confirmed their benefits including: lack of the discretization, the efficient and not cumbersome modeling of the boundary and the domain and their modification and the smaller number of data required to modeling, hence the systems of equations with smaller number of equations to solve. PIESs as previously mentioned numerical methods are also characterized by certain disadvantages. For this reason that they are the modification of classical BIEs we have to deal with the reduced accuracy of solutions and their derivatives in the vicinity of the boundary. Furthermore, there is the difficulty in calculating the derivatives of solutions on the boundary (such as stresses in elasticity problems), or even in the domain (such as stresses in plasticity problems), because it requires calculation of strongly singular integrals. Developing the strategy that at least partially eliminates these drawbacks may cause the possibility of extending the applications of PIESs.

The paper proposes the strategy for computing the derivatives of solutions, whose objective is to eliminate above-described problems. The strategy can replace methods known from BEM: analytical which gives the integral identity with strongly singular integrands or numerical via the classical approach to the numerical calculation of derivatives. The first approach is difficult to implement due to the necessity of the accurate calculation of singular integrals, whilst second is computationally expensive due to obtaining derivatives only at previously declared nodes. There are also other significant techniques that overcome the problem of the singular integration obtained after analytical differentiation of the integral identity for displacements. In the paper [21] the singularities involved in integration kernels are analytically removed by expressing the non-singular parts of the integration kernels as power series in the local distance of the intrinsic coordinate system.

A very efficient and versatile approach is to calculate derivatives by direct substitution of the point (its coordinates) to previously obtained expressions, which approximate the partial derivatives of the required order. A similar concept is used in the meshless methods, but as mentioned before there we have to deal with difficulties such as: posing boundary conditions or the computational complexity [15,16,8]. In order to avoid some of these problems one can apply e.g. interpolation, which allows for constructing the shape functions which possess Kronecker delta function property, as in the point interpolation method (PIM) [16]. There is also the problem with the reversibility of the so generated moment matrix, because of its singularity.

The main aim of this paper is to develop an effective approximation strategy for the derivatives of solutions of boundary value problems. Intermediate purpose is to use the advantages of PIES to resolve these problems and receive direct solutions in an efficient manner. Achieving the intended purpose requires obtaining expressions approximating the partial derivatives of any order. Various approaches to obtaining mentioned expressions are proposed and tested. For different approaches we develop different variants, and we test their impact on the accuracy and stability of the results. The concept is presented by elasticity problems, whilst its credibility is tested on several examples, and the results are compared with the analytical solutions.

2. Pies for 2D solids

2.1. Relations describing the linear elastic body

We consider the isotropic, homogeneous, linear, elastic body defined in a domain Ω in the two-dimensional Euclidean space with the boundary $\Gamma = \Gamma_u \cup \Gamma_p$ (Fig. 1).

The body is under the influence of body forces $\bar{b}(x,y) = (b_x, b_y)$ and tractions $\bar{p}(x,y) = (p_x, p_y)$, which cause the displacement field $\bar{u}(x,y) = (u_x, u_y)$, the strain field $\varepsilon_{ij}(x,y)$ and the stress field $\sigma_{ij}(x,y)$, $i,j = x,y$.