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Preconditioning analysis of nonuniform incremental unknowns method for two dimensional elliptic problems

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ABSTRACT

For the linear system obtained by discretizing two dimensional elliptic boundary value problems on nonuniform meshes, the condition number of the coefficient matrix preconditioned by nonuniform incremental unknowns (NUIUs) method, abbreviated as NUIUs matrix, is carefully analyzed. Comparing to the original coefficient matrix, the condition number of the NUIUs matrix is reduced from $O(a^d)$ to $O(d^2)$ with $a \geq 4$ and d being the level of discretization. Numerical experiments are performed, respectively, on three types of nonuniform meshes to verify the correctness of our theoretical analysis and test the preconditioning efficiency of the NUIUs method.

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1. Introduction

For two dimensional elliptic boundary value problems, we use finite differences on *uniform meshes* for the spatial discretization. The problem of finding the solutions of these equations reduces to approximate the solutions of large sparse systems of linear equations $AU = b$. Using the classical incremental unknowns (IUs) method to precondition these systems leads to a kind of new linear systems of the form

$$\bar{A}\bar{U} = \bar{b}, \tag{1.1}$$

where $\bar{A} = S^T AS$ and $\bar{b} = S^T b$. Here S stands for the transfer matrix from IUs vector $\bar{U}$ to nodal unknowns (NUs) vector U, i.e., $U = S\bar{U}$. Since IUs method on uniform mesh usually yields a very good conditioned matrix in linear algebraic equations [1–7], i.e., the condition number of IUs matrix $\bar{A}$ is much smaller than that of the NUs matrix A, we always solve linear system (1.1), instead of $AU = b$, with direct or iterative solution methods [8–11]. Due to the efficiency referred above, IUs method was introduced to solve a number of partial differential equations [12–22] including many classical problems of fluid dynamics [23–27].

For singularly perturbed differential equations, approximating the solution on uniform mesh is no more effective because of the appearance of boundary or interior layers in the solutions. One of the most promising strategies, for handling the boundary or interior layers and improving the accuracy of the numerical solution, is to discretize the equations on a suitable nonuniform mesh. The condition number of the NUs matrix, derived by discretizing the singularly perturbed problem on a

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d-level nonuniform mesh, is the order of $O(a^d)$ with $a \geqslant 4$ being a mesh-dependent real number, which is always greater than that of the coefficient matrix derived on uniform meshes $O(4^d)$. With the increases of the discretization level d, the condition number increases very fast, which means the workload for approximating the solution of linear system will escalate sharply. To reduce the condition number of the coefficient matrix, IUs method should be a good choice. However, the classical IUs method can not be directly used to precondition the corresponding linear system since the discretization mesh is no more uniform. This deficiency was overcome by Chehab and Miranville in [28] by generalizing IUs method to nonuniform meshes (we call it nonuniform IUs method or NUIUs method). The definition of NUIUs method can be described as follows:

Denote by $\mathscr{M}_d := \{M_{ij}^{(d)}\}_{i,j=0}^{2^d N}$ the nodal set on the region $\Omega = (0,1)^2$ with $M_{ij}^{(d)} = \left(x_i^{(d)}, y_j^{(d)}\right)$ satisfying

$$\begin{aligned} &0 = x_0^{(d)} < x_1^{(d)} < x_2^{(d)} < \ldots < x_i^{(d)} < x_{i+1}^{(d)} < \ldots < x_{2^d N-1}^{(d)} < x_{2^d N}^{(d)} = 1, \\ &0 = y_0^{(d)} < y_1^{(d)} < y_2^{(d)} < \ldots < y_j^{(d)} < y_{j+1}^{(d)} < \ldots < y_{2^d N-1}^{(d)} < y_{2^d N}^{(d)} = 1, \end{aligned} \tag{1.2}$$

where d is the level of discretization and N is the number of the coarsest meshes along directions x and y. Let $u_{ij}^{(d)} \approx u(M_{ij}^{(d)})$ be the approximate value of function u at nodal point $M_{ij}^{(d)}$. Then, the one-level NUIUs can be defined as

- At coarse nodal points $M_{2i,2j}^{(d)}$, with $i,\ j = 0, 1, \ldots, 2^{d-1}N$,

  $$y_{2i,2j}^{(d)} = u_{2i,2j}^{(d)},$$

- At fine nodal points $M_{2i,2j+1}^{(d)}$, with $i = 0, 1, \ldots, 2^{d-1}N$ and $j = 0, 1, \ldots, 2^{d-1}N - 1$,

  $$z_{2i,2j+1}^{(d)} = u_{2i,2j+1}^{(d)} - \left[\alpha_2 u_{2i,2j+2}^{(d)} + \alpha_1 u_{2i,2j}^{(d)}\right],$$

- At fine nodal points $M_{2i+1,2j}^{(d)}$, with $i = 0, 1, \ldots, 2^{d-1}N - 1$ and $j = 0, 1, \ldots, 2^{d-1}N$,

  $$z_{2i+1,2j}^{(d)} = u_{2i+1,2j}^{(d)} - \left[\beta_2 u_{2i+2,2j}^{(d)} + \beta_1 u_{2i,2j}^{(d)}\right],$$

- At fine nodal points $M_{2i+1,2j+1}^{(d)}$, with $i,\ j = 0, 1, \ldots, 2^{d-1}N - 1$,

  $$z_{2i+1,2j+1}^{(d)} = u_{2i+1,2j+1}^{(d)} - \left[\alpha_1\beta_1 u_{2i,2j}^{(d)} + \alpha_1\beta_2 u_{2i+2,2j}^{(d)} + \alpha_2\beta_1 u_{2i,2j+2}^{(d)} + \alpha_2\beta_2 u_{2i+2,2j+2}^{(d)}\right],$$

  where

$$\alpha_1 = \frac{y_{2j+2}^{(d)} - y_{2j+1}^{(d)}}{y_{2j+2}^{(d)} - y_{2j}^{(d)}}, \quad \alpha_2 = \frac{y_{2j+1}^{(d)} - y_{2j}^{(d)}}{y_{2j+2}^{(d)} - y_{2j}^{(d)}}, \quad \beta_1 = \frac{x_{2i+2}^{(d)} - x_{2i+1}^{(d)}}{x_{2i+2}^{(d)} - x_{2i}^{(d)}}, \quad \beta_2 = \frac{x_{2i+1}^{(d)} - x_{2i}^{(d)}}{x_{2i+2}^{(d)} - x_{2i}^{(d)}}. \tag{1.3}$$

The NUIUs at any level l with $1 \leqslant l \leqslant d$ can be defined recursively. One may refer to [6] for a similar definition of the multilevel block IUs.

From the numerical results presented by Chehab and Miranville in [28], the NUIUs method can significantly reduce the condition number of the coefficient matrix. However, up to now, there is no theoretical estimation for the preconditioning effect. In order to make up this deficiency and improve the mathematical theory of NUIUs method, the condition number of the d-level NUIUs matrix $\bar{A}_d$ for two dimensional elliptic boundary value problem is analyzed in this work. The analysis method can be easily generalized to more general elliptic and parabolic problems. Although the preconditioning effect of IUs method for linear elliptic problems are close to that of the well known hierarchical bases (HB) method in finite elements [29,30], the estimation of the condition number of the NUIUs matrix is still interesting and significant since it can also provide theoretical support for the application of NUIUs method in many practical problems, such as the simulation of complex flows, the solution of singular perturbed problems and so on.

The remainder part of this paper is organized as follows. In Section 2, the nonuniform meshes, subspaces and several useful norms associated with NUIUs method are introduced. The relationships of these useful norms are carefully studied in Section 3. In Section 4, we estimate the condition number $\kappa(\bar{A}_d)$ of the NUIUs coefficient matrix. Finally in Section 6, numerical experiments are performed on three types of nonuniform meshes. The numerical results support our theoretical analysis and verify the efficiency of NUIUs method.

2. Mathematical setting

Let $\Omega = (0,1)^2$ and $\mathscr{M}_l := \{M_{ij}^{(l)}\}_{i,j=0}^{2^l N}$ with $l = 0, 1, \ldots, d$ be the nodal sets defined on Ω with $M_{ij}^{(l)} = (x_i^{(l)}, y_j^{(l)})$ satisfying $x_i^{(l)} = x_{2^{d-l}i}^{(d)}$ and $y_j^{(l)} = y_{2^{d-l}j}^{(d)}$. Denote by

$$h_i^{(l)} = x_i^{(l)} - x_{i-1}^{(l)}, \quad r_j^{(l)} = y_j^{(l)} - y_{j-1}^{(l)}, \quad i,\ j = 1, 2, \ldots, 2^l N,$$

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