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Rational pseudospectral approximation to the solution of a nonlinear integro-differential equation arising in modeling of the population growth

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ABSTRACT

Pseudospectral approach based on rational Legendre and rational Chebyshev functions is developed to solve the nonlinear integro-differential Volterra's population model. The model includes an integral term that characterizes accumulated toxicity on the species in addition to the terms of the logistic equation. Since the equation is defined on positive real line, the rational Legendre and the rational Chebyshev functions are used to approximate the unknown function. The approach reduces the solution of the main problem to the solution of a system of nonlinear algebraic equations. The obtained results represent the exponential convergence of the new method, so it can be applied on a wide variety of problems.

1. Introduction

In this paper, we propose two numerical methods based on the pseudospectral approach. The methods have shown good performance for solving ordinary differential equations [1–4]. In the current paper, they are used to approximate the solution of the Volterra's population problem which is a nonlinear integro-differential equation. These methods use the rational Legendre and rational Chebyshev functions to approximate the unknown function $u(t)$ of the model.

Populations of organisms tend to increase as far as their environment will allow. As a result, most populations are in a dynamic state of equilibrium. Their numbers increase in a delicate balance that is influenced by limiting factors. Population dynamics of a specific species are determined by these underlying factors. These factors, in general, are nutritional components level, crowding and competition, and waste concentration increase [5].

Population dynamics express itself in different forms depending on what kind of species to which member of a population belong, sometimes, a population will grow suddenly in a population explosion that creates adverse environmental conditions. As a result, great numbers die suddenly, called a population crash. Taking the microbial growth curve, as our example, to be studied, we find by analyzing it, when microorganisms are cultivated in a batch culture or closed system, the resulting curve has four distinct phases: lag phase, exponential phase, stationary phase and death phase [5,6].

An interesting model for the interaction between a number of different biological species was introduced by Volterra. He was motivated to investigate competing species by discussions with his friend D'Ancona [7]. Also we refer the interested reader to [8,9] for more useful models with applications in mathematical biology and engineering [10].

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The Volterra's model for population growth of a species within a closed system is given in [11–13] as

$$\frac{\mathrm{d}p}{\mathrm{d}\tilde{t}} = ap - bp^2 - cp\int_0^{\tilde{t}} p(x)\mathrm{d}x, \qquad p(0) = p_0, \tag{1}$$

where $a > 0$ is the birth rate coefficient, $b > 0$ is the intraspecies competition, $c > 0$ is the toxicity coefficient, p_0 is the initial population, and $p = p(\tilde{t})$ denotes the population at time $\tilde{t}$. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long run. In the case $c = 0$, we have the well-known logistic equation. The last term contains the integral that indicates the "total metabolism" or total amount of toxins produced since time zero. The individual death rate is proportional to this integral, and so the population death rate due to toxicity must include a factor u. The presence of the toxic term due to the system being closed always causes the population level to fall to zero in the long run. The relative size of the sensitivity to toxins, c, determines the manner in which the population evolves before its fated decay [12,14].

By applying the non-dimensional variables

$$t = \frac{\tilde{t}c}{b}, \quad u = \frac{pb}{a}, \tag{2}$$

the time and the population can be scaled. This leads to non-dimensional problem

$$\kappa\frac{\mathrm{d}u}{\mathrm{d}t} = u - u^2 - u\int_0^t u(x)\mathrm{d}x, \qquad u(0) = u_0, \tag{3}$$

where $u(t)$ is the scaled population of identical individuals at time t, and $\kappa = \frac{c}{ab}$ is a prescribed non-dimensional parameter. This is a parabolic Volterra problem [15].

The analytical solution [12]

$$u(t) = u_0 \exp\left(\frac{1}{\kappa}\int_0^t \left[1 - u(\tau) - \int_0^\tau u(x)\mathrm{d}x\right]\mathrm{d}\tau\right), \tag{4}$$

shows that $u(t) > 0$ for all t, if $u_0 > 0$.

In [12,14], solutions of the Volterra's population equation are considered by means of the singular perturbation techniques. In order to establish the size of the solution and the time scale, authors of [12,14] scaled out the parameters of the equation as much as possible. There are four different ways to do scaling. These authors considered two cases: small κ and large κ, and showed that for small κ, where the population is relatively insensitive to toxins, the population $u(t)$ increases rapidly before reaching a peak according to logistic behavior, before decaying exponentially slowly to zero. Furthermore, it is shown that if κ is large, where the populations are extremely sensitive to toxins, the solution $u(t)$ is proportional to $\mathrm{sech}^2(t)$ and has a smaller amplitude. Scudo [11], offered the method of successive approximations for the solution of Volterra's population equation, but he did not implement it. In [13] several methods for solving the non-dimensional Volterra's equation are presented. Author of [13] implemented two approaches to numerically solve this equation for arbitrary κ. He also transformed the equation into a coupled system of two first-order ODEs and performed a phase-plane analysis for arbitrary κ. The correlations among the various results are made. In [16], the author implemented the series solution and the decomposition methods independently to solve the Volterra's population model and its converted form to nonlinear ODE. He used Padé approximations [17] in the analysis to capture the essential behavior of the population $u(t)$. The author of [5] applied the Adomian decomposition [18] and Sinc–Galerkin methods to the solution of Volterra's population model. He compared these methods and showed that the Adomian decomposition method is more efficient and easy to use. A numerical algorithm for approximate solutions of a fractional population growth model in a closed system is given in [19]. The algorithm is based on Adomian decomposition approach and the solutions are calculated in the form of a convergent series. The authors showed by using the Padé approximations [17] the model increases rapidly along the logistic curve followed by a slow exponential decay after reaching a maximum point. In [20–28], the authors converted the Volterra's population model to an ODE and then applied the methods to solve the ODE. Authors of [20,21] applied tau method based on rational functions to solve the Volterra's population ODE. In [22], He's homotopy perturbation method [29] is proposed for solving Volterra's ODE. The solution of equation is approximated by using the differential transform method. He's homotopy perturbation method is implemented to the model and the solution is obtained. In [23] the properties of composite spectral functions consisting of few terms of orthogonal functions are presented and are utilized to reduce the solution of the Volterra's ODE to the solution of a system of algebraic equations. Authors of [24] applied the second derivative of solution in algorithm of multistep methods and introduced a new class of methods which are known as the second derivative multi-step methods. They used the method to solve the Volterra's population model. Authors of [25] proposed a numerical method based on hybrid function approximations to solve the Volterra's ODE model. They also presented the properties of the hybrid functions that consist of block-pulse and Lagrange-interpolating polynomials. A comparison between rational Chebyshev and Hermite collocation methods for solving Volterra's population ODE is presented in [26]. In [27], the quasilinearization method, which is