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Estimation of some lifetime parameters of generalized Gompertz distribution under progressively type-II censored data

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A R T I C L E I N F O

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A B S T R A C T

In this paper, the estimation of parameters of a three-parameter generalized Gompertz distribution based on progressively type-II right censored sample is studied. These methods include the maximum likelihood estimators (MLEs), and Bayesian estimators. Approximate confidence intervals for the unknown parameters as well as reliability function, hazard function and coefficient of variation are constructed based on the s-normal approximation to the asymptotic distribution of MLEs, and log-transformed MLEs. Furthermore, two bootstrap confidence intervals are proposed. Several Bayesian estimates are obtained against different symmetric and asymmetric loss functions such as squared error, LINEX and general entropy. Based on theses loss functions, under gamma priors distributions, Bayes estimates of the unknown parameters and the corresponding credible intervals are obtained by using the Gibbs within Metropolis–Hasting samplers procedure. Finally, a real data set is analyzed to illustrate the proposed methods.

1. Introduction

The bathtub-shape hazard function provides an appropriate conceptual model for some electronic and mechanical products as well as the lifetime of humans. In recent years, some probability distributions have been proposed to fit real life data with bathtub shape failure rates. For example, [1] proposed a three-parameter distribution with increasing, decreasing, constant or bathtub-shaped failure rate function. A three-parameter model, called exponentiated Weibull distribution, was introduced by [2]. Another three-parameter model was proposed by [3] and called extended Weibull distribution. [4] addressed the extended Weibull distribution in [3] by introducing an additional parameter to an existing model and also investigated some properties of this distribution and studied the versatility of the model in modeling failure data. [5] proposed a three-parameter modified Weibull extension with a bathtub shaped hazard function. Another generalization of the Weibull distribution is the modified Weibull distribution discussed in [6]. Besides, Chen proposed his new model in his paper ([7]). An earlier review can be found in [8] and some new models for lifetime data have been proposed very recently such as in [9–12].

Gompertz distribution is applicable in the theory of extreme-order statistics, gives a good fit to data from clinical trials on order subjects, and is useful in the construction of life tables. Also it is used in medical science (see [13]). It has been recognized as a useful model for the analysis of lifetime data. In this article, we focus on a three-parameter generalized Gompertz with the bathtub shape or increasing failure rate function proposed by [12].

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The random variable X has a generalized Gompertz distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$f_X(x) = \theta\beta e^{\lambda x} e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)} \left[1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]^{\theta-1}, \quad x > 0, \quad \beta > 0, \quad \theta > 0, \quad \lambda \geqslant 0, \tag{1}$$

and

$$F_X(x) = \left[1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]\theta, \quad x > 0, \quad \beta > 0, \quad \theta > 0, \quad \lambda \geqslant 0. \tag{2}$$

The generalized Gompertz distribution with parameters β, λ and θ will be denoted by GGD(β, λ, θ). Its reliability and hazard functions are given by

$$S(t) = 1 - F(t) \quad \text{and} \quad H(t) = f(t)/S(t), \quad t > 0. \tag{3}$$

The GGD family not only covers the one-parameter exponential distribution (λ tends to zero and θ equals one) and generalized exponential distribution (λ tends to zero) as sub-distributions, but also covers the two parameter Gompertz distribution (θ equals one) as a special sub-family distribution. One of the good properties of GG distribution is that it can have different types of hazard rate shapes so that it can be applied to different kinds of products. It is clear that the shape of hazard rate function depends solely on the shape parameters λ and θ. Therefore, when $\lambda > 0$ and $\theta > 1$, the failure rate exhibits an increasing failure rate function curve; when $\lambda = 0, \theta < 1$, failure rate has a decreasing failure rate function and failure rate has a bathtub shaped failure rate when $\lambda > 0$, $\theta < 1$.

The coefficient of variation is used in numerous areas of science such as biology, economics, and psychology, and in engineering in queueing and reliability theory (see, for example, [14]). [15] gave a summary of uses of the coefficient of variation in a number of areas. Given a set of observations from GGD(β, λ, θ), the sample coefficient of variation (CV) is often estimated by the ratio of the sample standard deviation to the sample mean. Or equivalent

$$CV = \frac{\sqrt{E(X^2) - (E(X))^2}}{E(X)} = G(\beta, \lambda, \theta), \tag{4}$$

where $E(X)$ and $E(X^2)$ are the first and the second moments of the GGD(β, λ, θ), given by

$$\begin{aligned} E(X) &= \theta\beta e^{\frac{\beta}{\lambda}} \int_0^\infty x e^{\lambda x} e^{-\frac{\beta}{\lambda}e^{\lambda x}} \left[1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x}-1)}\right]^{\theta-1} dx = \theta\beta \sum_{j=0}^\infty \binom{\theta-1}{j} (-1)^j e^{\frac{\beta}{\lambda}(j+1)} \int_0^\infty x \frac{-1}{\beta(j+1)} \{-\beta(j+1)e^{\lambda x}\} e^{-\frac{\beta}{\lambda}(j+1)e^{\lambda x}} dx \\ &= \theta \sum_{j=0}^\infty \binom{\theta-1}{j} \frac{(-1)^j e^{\frac{\beta}{\lambda}(j+1)}}{(j+1)} \int_0^\infty e^{-\frac{\beta}{\lambda}(j+1)e^{\lambda x}} dx = \frac{\theta}{\lambda} \sum_{j=0}^\infty \binom{\theta-1}{j} \frac{(-1)^j e^{\frac{\beta}{\lambda}(j+1)}}{(j+1)} \int_{(j+1)}^{\frac{\beta}{\lambda}\infty} u^{-1} e^{-u} du \\ &= \frac{\theta}{\lambda} \sum_{j=0}^\infty \binom{\theta-1}{j} \frac{(-1)^j e^{\frac{\beta}{\lambda}(j+1)}}{(j+1)} \Gamma\left[0, (j+1)\frac{\beta}{\lambda}\right], \end{aligned} \tag{5}$$

and

$$\begin{aligned} E(X^2) &= \theta\beta \sum_{j=0}^\infty \binom{\theta-1}{j} (-1)^j e^{\frac{\beta}{\lambda}(j+1)} \int_0^\infty x^2 e^{\lambda x} e^{-\frac{\beta}{\lambda}(j+1)e^{\lambda x}} dx = \frac{\theta}{\lambda^2} \sum_{j=0}^\infty \binom{\theta-1}{j} \frac{(-1)^j e^{\frac{\beta}{\lambda}(j+1)}}{(j+1)} \int_{(j+1)}^{\frac{\beta}{\lambda}\infty} \ln^2\left[\frac{\lambda u}{\beta(j+1)}\right] u^{-1} e^{-u} du \\ &= \frac{\theta}{\lambda^2} \sum_{j=0}^\infty \binom{\theta-1}{j} \frac{(-1)^j e^{\frac{\beta}{\lambda}(j+1)}}{(j+1)} \left\{\gamma^2 + \frac{\pi}{6} - \frac{2\beta(j+1)}{\lambda}\ {}_3F_3\left[\{1,1,1\},\{2,2,2\}, -\frac{\beta(j+1)}{\lambda}\right] \right. \\ &\left. -2\gamma \ln\left(\frac{\lambda}{\beta(j+1)}\right) + \ln^2\left[\frac{\lambda}{\beta(j+1)}\right]\right\}, \end{aligned} \tag{6}$$

where $\Gamma[u, v]$ is incomplete gamma function, define by $\Gamma[u, v] = \int_v^\infty x^{u-1} e^{-x} dx, \gamma$ is Euler constant gamma with numerical value 0.577216, and ${}_pF_q\ [\{a_1, a_2, \ldots, a_p\}, \{b_1, b_2, \ldots, b_q\}, z]$ is a generalized hypergeometric function has series expansion

$${}_pF_q[\{a_1, a_2, \ldots, a_p\}, \{b_1, b_2, \ldots, b_q\}, z] = 1 + \sum_{k=1}^\infty \left[\frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \cdot \frac{z^k}{k!}\right],$$

where $(a)_k = a(a+1)\ldots(a+k-1)$.

[12] discussed statistical properties for the GGD, specially moments, modes, quantiles and median. Also, they investigated the maximum likelihood estimators of the unknown parameters and its asymptotic confidence intervals based on complete data. In industrial life testing and medical survival analysis, very often the object of interest is lost or withdrawn before failure or the object lifetime is only known within an interval. Hence, the obtained sample is called a censored sample (or an incomplete sample). Some of the major reasons for removal of the experimental units are saving the working experimental units for future use, reducing the total time on test and lower the cost associated with these. Right censoring is one of the censoring techniques used in life-testing experiments. The most common right censoring schemes are type-I