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# Synchronization for complex networks with Markov switching via matrix measure approach ☆

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ABSTRACT

This paper devotes to almost sure synchronization and almost sure quasi-synchronization of complex networks with Markov switching. Some sufficient conditions are derived in terms of the ergodic theory of continuous time Markov chain and the matrix measure approach, which can guarantee that the dynamical networks almost surely synchronize or quasi-synchronize to a given manifold. According to the property of Markov chain and the exponential distribution of switching time sequence, we also estimate the probability distribution of the quasi-synchronization error for a two-state Markov chain and then generalize them to a finite state space Markov chain. Meanwhile, some examples with numerical simulations are given to show that the Markov chain plays an important role in synchronization of networks.



## 1. Introduction

In recent years, complex dynamical networks have attracted a lot of attention in science and engineering [1–3]. The synchronization of all dynamical nodes is an important and interesting phenomena mostly because the synchronization can well explain many natural phenomena. Therefore, the synchronization of dynamical networks has been actively studied due to its wide applications for physics, communication, etc. Recently, there has been an increasing interest in the study of synchronization of complex dynamical networks. In the previous works, Pecora and Carroll [4] showed that coupled chaotic systems can be synchronized in 1990. Hereafter they introduced master stability function method [5] to study the local synchronization of coupled chaotic systems. By Lyapunov direct method, some sufficient conditions for synchronization in an array of linearly coupled dynamical systems were proposed in [6–8]. Such sufficient conditions depended on the second smallest eigenvalue of the Laplacian matrix in case that the graph was undirected. After this work, many control schemes such as adaptive control [9,10], pinning control [11,12], fuzzy control [13], impulsive control [14–16], and intermittent control [17–19] are widely applied to achieve synchronization of complex dynamical networks.

Another important subject for the emergent behavior in dynamical networks is consensus in multi-agent systems. Indeed, consensus in multi-agent systems, which implies that all the agents will reach an agreement in a certain manner, is a special

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case of synchronization. Recently, there have been a lot of researches on consensus in multi-agent systems, which can be referred to [20–22] for details and references. It is well known that random uncertainties widely exist in biological networks due to environmental noise (e.g. white noise and color noise). Such systems are described by stochastic differential systems which have been used efficiently in modeling many practical problems that arise in the fields of engineering, physics, and economics as well. So the theory of stochastic differential equation is attracting much attention in recent years [23–25]. Based on the theory of stochastic differential equations, a lot of synchronization results of dynamical networks or consensus in multi-agent systems with white noise have been obtained [26–28]. However, telegraph noise, which is a simple color noise, can be illustrated as a switching between two or more regimes of environment. If the switching is memoryless and the waiting time for the next switch has an exponential distribution, then we can model the regime switching by a finite-state Markov chain. In [29], Padilla and Adolph present a mathematical model for predicting the expected fitness of phenotypically plastic organisms experiencing a variable environment and discussed the importance of time delays in this mode. The authors [30] discussed the effect of telegraph noise on the well known SIS epidemic model and established the condition for extinction and persistence for the SIS epidemic model with Markov switching. In 1958, Hajnal [31] investigated the weak ergodicity of non-homogenous Markov chains and proposed scrambling matrix, which plays an important role in the convergence of products of stochastic matrices. Based on the work of Hajnal, Salehi and Jadbabaie [32] provided a necessary and sufficient condition for convergence of consensus algorithms when the underlying graphs of the network are generated by an ergodic and stationary random process. In [33], complex networks with stochastically switching coupling structures was investigated. Stochastic switching coupling networks are addressed by independent and identically distributed switching processes or Markov jump processes. Some other researches on synchronization or consensus of networks with stochastically switching topology can be referred in [34–36].

The aim of this paper is to study almost sure synchronization and almost sure quasi-synchronization of complex dynamical networks with Markov chain taking value in finite state space. It is assumed that switching time sequence follows the exponential distribution, which means that we do not require the networks to switch fast enough [37], also we do not require the networks with the same expect of switching time interval [33]. By using ergodic theory of continuous time Markov chain [38] and the matrix measure approach [39,40], some sufficient condition is derived to ensure that the switched networks almost surely synchronize to the switched manifold. It is interested that if subsystems are not synchronized, but the other subsystems are synchronized, then the over system will achieve synchronization in the end. This shows that Markov chain plays the important role in the synchronous behavior of networks. In addition, if the synchronization manifold is a chaotic system without switching structure, parameter mismatches [41,42] are unavoidable in the implementations of chaos synchronization systems. In this sense, we investigate almost sure quasi-synchronization between the switched networks and the chaotic system. We also estimate the probability distribution of the quasi-synchronization error by the property of Markov chain and the exponential distribution for switching time interval. Finally, some examples with numerical simulations are given to illustrate the applicability of the results. The rest of this paper is organized as follows. In Section 2, complex networks model with Markov switching is presented, together with some lemmas of solution. In Section 3, almost sure synchronization is derived for switching networks by ergodic theory of continuous time Markov chain. Section 4 devotes to the investigation of almost sure quasi-synchronization of networks and the probability distribution of quasi-synchronization error. In Section 5, some numerical examples are given to demonstrate that our results. At last, some conclusions are given in Section 6.

**Notations:** Throughout this paper, $R = (-\infty, +\infty), R^+ = [0, +\infty), R^n$ denotes the $n$-dimensional Euclidean space. the superscript $T$ represents the transpose. $I_N$ stands for the identity matrix with $N$ dimension.

## 2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual condition (i.e., it is increasing right continuous and $\mathcal{F}_0$ contains all $P$-null sets), $r(t), t \geqslant 0$ be a right continuous Markov chain on the probability space taking values in the state space $\mathbb{S} = \{1, 2, \ldots, M\}$ with generator $\Gamma = (\delta_{ij})_{M\times M}$, where $\delta_{ii} = -\sum_{1\leqslant j\leqslant M, j\neq i}\delta_{ij}$ and $\delta_{ij} > 0 (i \neq j)$ is the transition rate from $i$ to $j$, that is $P\{r(t+\delta) = j | r(t) = i\} = \delta_{ij}\epsilon + o(\epsilon)$, where $\epsilon > 0$. By [38], we see that almost all sample paths of $r(t)$ are constants except for a finite number of jumps in any finite subinterval of $[0, \infty)$. Moreover, there is a sequence $\{\tau_k\}_{k\geqslant 0}$ of finite valued $\mathcal{F}_t$ stopping times such that $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ almost surely and

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k)\mathcal{I}_{[\tau_k,\tau_{k+1})}(t), \tag{2.1}$$

where $\mathcal{I}_A$ denotes the indicator function of set $A$. Given that $r(\tau_k) = i$, the random variable $\tau_{k+1} - \tau_k$ follows the exponential distribution with parameter $-\delta_{ii}$. That is

$$P(\tau_{k+1} = j | \tau_k = i) = -\frac{\delta_{ij}}{\delta_{ii}}, \quad j \neq i, \quad P(\tau_{k+1} - \tau_k \geqslant t | r(\tau_k) = i) = e^{\delta_{ii} t}, \quad \forall t \geqslant 0.$$

Furthermore, this markov chain has a unique stationary distribution $\Pi = (\pi_1, \pi_2, \ldots, \pi_M)$ satisfying $\Pi\Gamma = 0$ and $\sum_{i=1}^{M}\pi_i = 1$. We consider the following coupled network with Markovian switching