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Short communication

Logistic function as solution of many nonlinear differential equations

Nikolay A. Kudryashov

Department of Applied Mathematics, National Research Nuclear University "MEPhI", 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

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ABSTRACT

The logistic function is shown to be solution of the Riccati equation, some second-order nonlinear ordinary differential equations and many third-order nonlinear ordinary differential equations. The list of the differential equations having solution in the form of the logistic function is presented. The simple method of finding exact solutions of nonlinear partial differential equations (PDEs) is introduced. The essence of the method is based on comparison of nonlinear differential equations obtained from PDEs with standard differential equations having solution in the form of the logistic function. The wide application of the logistic function for finding exact solutions of nonlinear differential equations is demonstrated.

1. Introduction

The logistic function (the sigmoid function) is determined by the following formula [1–4]

$$Q(z) = \frac{1}{1+e^{-z}}, \tag{1.1}$$

where z is independent variable on the complex plane. We see that the logistic function has the pole of the first order on complex plane. This function can be used for finding exact solutions of nonlinear differential equations [4–6]. Other variants of this approach without the logistic function were used in some papers before (see, for a example [7–11]).

One can see that the logistic function is the solution of the first order differential equation called the Riccati equation [4–6]

$$Q_z - Q + Q^2 = 0. \tag{1.2}$$

The logistic function (1.1) can be presented taking the hyperbolic tangent into account because of the following formula

$$\frac{1}{1+e^{-z}} = \frac{1}{2}\tanh\left(\frac{z}{2}\right) + \frac{1}{2}. \tag{1.3}$$

However the logistic function is more convenient for finding exact solutions as it has been illustrated in recent papers [4–6].

Let us show that the general solution of the Riccati equation can be expressed via the logistic function. The Riccati equation takes the form

E-mail address: nakudr@gmail.com

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$$y_z = a y^2 + b y + c, \tag{1.4}$$

where a, b and c are arbitrary constants.

It is easy to obtain that the general solution of Eq. (1.4) can be written by means of formula

$$y = B - \frac{2B + b}{a} Q(z) \quad z = \frac{z' - z_0}{2B + b}, \tag{1.5}$$

where z_0 is an arbitrary constant, B is defined via constants a, b and c from the algebraic equation

$$a B^2 + b B + c = 0. \tag{1.6}$$

So, the logistic function is the solution of the Riccati equation to within transformations (1.5).

The aim of this paper is to find some nonlinear ordinary differential equations of the second and the third order with exact solutions in the form of the logistic function and to show that there are nonlinear partial differential equations having solution in the form of the logistic function. We also illustrate that one can find exact solutions of many nonlinear partial differential equations using the list of standard nonlinear ordinary differential equations.

2. Nonlinear ordinary differential equation of the second order with solution in form of logistic function

Differentiating Eq. (1.2) with respect to z we have the following second-order differential equation

$$Q_{zz} - Q_z + 2 Q Q_z = 0. \tag{2.1}$$

It is obvious that the logistic function $Q(z)$ satisfies Eq. (2.1) as well. At that time if we use the equality

$$Q_z = Q - Q^2, \tag{2.2}$$

we obtain three other differential equation

$$Q_{zz} - Q + Q^2 + 2 Q Q_z = 0, \tag{2.3}$$

$$Q_{zz} - Q_z + 2 Q^2 - 2 Q^3 = 0, \tag{2.4}$$

$$Q_{zz} - Q + 3 Q^2 - 2 Q^3 = 0 \tag{2.5}$$

having solutions in the form of the logistic function.

Taking into account these equations we can present the other second order nonlinear ordinary differential equations with solutions in the form of logistic function. These equations take the form

$$Q_{zz} - Q_z + 2 Q Q_z + F_1(Q, Q_z, \ldots)(Q_z - Q + Q^2) = 0, \tag{2.6}$$

$$Q_{zz} - Q + Q^2 + 2 Q Q_z + F_2(Q, Q_z, \ldots)(Q_z - Q + Q^2) = 0, \tag{2.7}$$

$$Q_{zz} - Q_z + 2 Q^2 - 2 Q^3 + F_3(Q, Q_z, \ldots)(Q_z - Q + Q^2) = 0, \tag{2.8}$$

$$Q_{zz} - Q + 3 Q^2 - 2 Q^3 + F_4(Q, Q_z, \ldots)(Q_z - Q + Q^2) = 0, \tag{2.9}$$

where $F_j(Q, Q_z, \ldots), \quad j = 1, \ldots, 4$ are some dependencies on Q, Q_z and so on.

Let us call Eqs. (2.6)–(2.9) as the standard nonlinear differential equations.

In the present time there are a lot of different methods for finding exact solutions of nonlinear differential equations. Here we only call some of them: the singular manifold method [7,8] that was modified by Kudryashov [5,4,12,13], the simplest equation method and some its modifications [9–11,14–17], the tanh-method [18–23], the G'/G – expansion method [24–26], the method for finding meromorphic solutions of nonlinear differential equations [27–29].

However we can look for exact solutions of many nonlinear partial differential equations taking into account the list of standard Eqs. (2.6)–(2.9) using the following simple algorithm. Let us take the nonlinear partial differential equation with the solution of the first order pole

$$E_1(u, u_t, u_x, u_{xx}, \ldots) = 0. \tag{2.10}$$

Using the traveling wave solution

$$u(x,t) = y(z), \quad z = kx - \omega t - k x_0, \tag{2.11}$$

we have the nonlinear ordinary differential equation in the form

$$E_2(y, -\omega y_z, k y_z, k^2 y_{zz}, \ldots) = 0. \tag{2.12}$$