Applied Mathematical Modelling 39 (2015) 5047–5057

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Gauss pseudospectral and continuation methods for solving two-point boundary value problems in optimal control theory

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ARTICLE INFO

Article history:
Received 16 March 2013
Received in revised form 15 March 2015
Accepted 1 April 2015
Available online 23 April 2015

Keywords:
Continuation method
Gauss pseudospectral method
Legendre–Gauss points
Optimal control
Pontryagin's minimum principle
Two-point boundary value problem

ABSTRACT

In this study, we propose an efficient pseudospectral method for solving two-point boundary value problems in optimal control theory. In our proposed approach, the Gauss pseudospectral method is utilized to reduce a two-point boundary value problem into the solution of a system of algebraic equations. However, the convergence to the solution of the system of equations obtained may be slow, or it can even fail, if a very good initial estimate of the optimal solution is not available. To overcome this drawback, we employ a numerical continuation method, which resolves the sensitivity of the proposed method to the initial estimate. The main advantages of the present combined method are that good results are obtained even when using a small number of discretization points, while the sensitivity to the initial estimate when solving the final system of algebraic equations is resolved successfully. The proposed method is especially useful when shooting methods fail due to the sensitivity or stiffness of the problem. We present numerical results for two examples to demonstrate the efficiency of the combined method.

1. Introduction

In recent decades, various techniques have been developed for efficiently solving two-point boundary value problems (TPBVPs), especially those in optimal control theory, such as the most popular shooting and multiple shooting methods [1,2]. However, in some real-life examples, these methods lead to numerically sensitive systems of algebraic equations [3] and other numerical methods are required to solve them.

In the present study, we propose a method for efficiently solving TPBVPs derived from the first order necessary conditions for optimality in the optimal control problems, where we also resolve the sensitivity of the proposed method to the initial estimate of the solution for the final algebraic system. To address this issue, we combine pseudospectral and continuation methods to derive a unified method for efficiently solving TPBVPs.

Pseudospectral methods are powerful tools for obtaining numerical solutions to practical engineering problems [4–7]. In particular, pseudospectral methods are usually the best tool for solving ordinary or partial differential equations with high accuracy in a simple domain if the data that define the problem are smooth [8]. In pseudospectral methods, two basic steps are required to obtain a numerical approximation of a solution for differential equations. First, an appropriate finite or discrete representation of the solution must be selected by polynomial interpolation of the solution based on some suitable

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http://dx.doi.org/10.1016/j.apm.2015.04.009
0307-904X/

points (nodes), such as Legendre–Gauss–Lobatto or Legendre–Gauss (LG) points [9]. The second step involves obtaining a system of algebraic equations by discretizing the original equation in the collocation points [10]. Recently, a special type of pseudospectral method called the Gauss pseudospectral (GP) method has been applied successfully to direct trajectory optimization problems [11–13]. In the GP method, the LG points are treated as collocation points that are not included among the end points. Thus, GP method is more suitable and readily applicable than traditional pseudospectral methods for discretizing TPBVPs.

In the first stage of our proposed method we apply GP method to reduce the solution of TPBVPs to the solution of a system of algebraic equations. This set of equations is then solved to obtain the values of unknown functions at the LG points. It should be noted that our method is similar to that of [14], except the approach in [14] employs the GP method to obtain a direct solution [3,15,16] and costate estimation [17] in optimal control problems. However, the system of algebraic equations that is obtained may be sensitive to the initial estimate. Thus, the convergence of the methods for obtaining solutions, such as Newton-like methods, requires a very good initial estimate, or a high level of user experience. Therefore, in the second stage, the continuation method is applied to address the sensitivity of the proposed method to the initial estimate of the solution for the final algebraic system.

Continuation methods have been applied as useful tools for solving a wide variety of problems [18–23]. The principal idea of the continuation method is to define a one-parameter family of problems such that the first problem is an easy problem to solve and the last is the original difficult problem. Each intermediate problem then uses the solution of the previous problem as an initial estimate in the family and this process continues until the solution of the original problem is achieved.

The remainder of this paper is organized as follows. In Section 2, we introduce the formulation of the TPBVPs. In Section 3, we give some preliminary details. In Section 4, we explain how the proposed method is applied to solve a TPBVP to obtain a set of algebraic equations. We then consider the solution of the problem. In Section 5, we give the formulation of the continuation method. In Section 6, the combined method is applied to two examples in optimal control theory to demonstrate the advantages of our method.

2. Problem statement

In this study, we are interested in the following class of optimal control problems. The problem involves finding a control vector $\mathbf{u}(t) = [u_1(t), \ldots, u_q(t)]^T$, the corresponding state vector $\mathbf{x}(t) = [x_1(t), \ldots, x_p(t)]^T$, and possibly the terminal time t_f that minimizes the functional

$$J = \Phi(\mathbf{x}(t_f), t_f) + \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)dt,$$

subject to a system of p nonlinear differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),$$

with the boundary conditions

$$\psi(\mathbf{x}(0), \mathbf{x}(t_f)) = \mathbf{0}, \tag{1}$$

where $\Phi : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$ is the terminal cost, which is also assumed to be smooth, $\psi : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^r, 0 \leqslant r \leqslant 2p$ and $g : \mathbb{R}^{p+q+1} \to \mathbb{R}$. The state $\mathbf{x}$ is continuous and the vector function $\mathbf{f} : \mathbb{R}^{p+q+1} \to \mathbb{R}^p$ is assumed to be a smooth function of the variables $(\mathbf{x}, \mathbf{u}, t)$.

To formulate the first order necessary conditions of optimality, the Hamiltonian function is introduced as follows

$$\mathscr{H}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}) = g(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{2}$$

where the function $\boldsymbol{\lambda}(t) = [\lambda_1(t), \ldots, \lambda_p(t)]^T$ is called the costate function. According to Pontryagin's minimum principle [24], the solution of the optimal control problem satisfies the following necessary conditions:

$$\dot{\mathbf{x}} = [\partial \mathscr{H} / \partial \boldsymbol{\lambda}]^T, \tag{3}$$

$$\dot{\boldsymbol{\lambda}} = -[\partial \mathscr{H} / \partial \mathbf{x}]^T, \tag{4}$$

$$0 = [\partial \mathscr{H} / \partial \mathbf{u}]^T. \tag{5}$$

By obtaining the control function $\mathbf{u}$ from Eq. (5) and replacing it in Eqs. (3) and (4), we obtain a system of differential equations based on $\mathbf{x}(t)$ and $\boldsymbol{\lambda}(t)$. We can express this system of differential equations as

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \tag{6}$$

where $\mathbf{y}(t) = [\mathbf{x}(t), \boldsymbol{\lambda}(t)]^T \in \mathbb{R}^{p+q}$, and $\mathbf{f} : \mathbb{R} \times \mathbb{R}^{p+q} \to \mathbb{R}^{p+q}$.

Furthermore, the boundary conditions (1) are also considered by Eq. (6). It should be noted that when the final time t_f is free, then the condition

$$\mathscr{H}(\mathbf{x}(t_f), \boldsymbol{\lambda}(t_f), \mathbf{u}(t_f)) + \frac{\partial \Phi}{\partial t}(\mathbf{x}(t_f), t_f) = 0, \tag{7}$$