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# Solutions of fuzzy LR algebraic linear systems using linear programs

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ABSTRACT

Using linear programs, we give the necessary and sufficient conditions for the solvability of a fuzzy LR algebraic linear system. Fuzzy LR algebraic linear systems are linear systems with LR fuzzy coefficient matrix and an LR fuzzy vector as the right-hand side. We consider an approximate solution when the fuzzy LR algebraic linear system lacks a solution. Also, we propose a linear programming problem to compute a solution or an approximate solution of a fuzzy LR algebraic linear system. Finally, by numerical examples, we show the suitability of our proposed approximate solution in comparison with another proposed approximate solution.



## 1. Introduction

Solving a fuzzy linear system has been popular during the past two decades (see [1,2]). Fuzzy linear systems have been studied under various assumptions. Buckley et al. [1,3] studied square fuzzy linear systems, $\tilde{A}\tilde{x} = \tilde{b}$, based on $\alpha$-cuts, extension principal and interval arithmetic, where the elements of $\tilde{A}$ and $\tilde{b}$ are fuzzy numbers. Several authors discussed various approaches for this fuzzy linear system or general forms of it (see [4–9]).

Another class of fuzzy linear systems includes fuzzy linear systems with the crisp coefficient matrix $A \in \mathbb{R}^{m\times n}$ and fuzzy vectors $\tilde{b}_{m\times 1}$ and $\tilde{x}_{n\times 1}$. Several authors studied such type of fuzzy linear systems under different hypotheses (e.g., see [10–18]).

Ghanbari et al. [19], studied $A\tilde{x} = \tilde{b}$, where $\tilde{x}$ and $\tilde{b}$ are LR fuzzy vectors and $A$ is an arbitrary crisp matrix. We showed that for solving a fuzzy LR linear system (when $\tilde{x}$ was an LR fuzzy vector), we must solve a crisp linear system and a constrained least squares problem, simultaneously. Then, when $A\tilde{x} = \tilde{b}$ lacked an exact solution, we introduced a new concept of an approximate solution. This new concept relied on solutions of the least squares models. To compute either exact or approximate solutions, we used an ABS algorithm. However, in [19], we considered that the corresponding crisp system was compatible (had a solution). In [20], Ghanbari and Mahdavi-Amiri removed this limitation. In other words, the necessity of the solvability of the corresponding crisp linear system was not needed. Thus, we proposed a more general concept of approximate solution based on a least squares model. To compute an approximate solution in [20], we proposed a quadratic programming model with bound constraints. Thus, in [20], we solved only an optimization problem, but in [19], we used to solve a crisp linear system and a constrained least squares problem, simultaneously. Also, in [20], we gave an example and a comprehensive numerical study to show the proposed approach in [20] was able to find better solutions than the ones obtained by our approach in [19]. In addition, in [20], we gave some new necessary and sufficient conditions for the solvability of $A\tilde{x} = \tilde{b}$.

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Using ranking functions, Ghanbari and Mahdavi-Amiri [21] proposed a new compromised solution of a fuzzy LR linear system, $A\tilde{x} = \tilde{b}$. Thus, this compromised solution was completely different from the solutions based on the least squares models proposed in [19] and extended in [20]. Also, in [21], we proposed a class of algorithms, based on ABS class of algorithms, to compute the general compromised solution when the ranking function was a member of a certain class of ranking functions. This general solution could be used for solving some fuzzy optimization problems.

One major class of fuzzy linear systems includes fuzzy linear systems with fuzzy coefficient matrix and fuzzy vector as the right-hand side ($\tilde{A}x = \tilde{b}$). Usually, authors name such fuzzy linear system, fuzzy algebraic linear system. This class of fuzzy linear systems arise in many fuzzy optimization problems (e.g., see [22–31]). Amirfakhrian [32] considered a fuzzy algebraic linear system using a normal system to obtain an approximate solution of fuzzy algebraic linear systems. Also, in [33], an iterative Gauss–Newton method was proposed to solve algebraic fuzzy equations with crisp coefficients. In addition, reader can find some numerical methods in [34,35].

From another perspective, optimization methods are used to solve fuzzy linear systems (e.g. see [8,10,21,19]). Here, we study LR type of fuzzy algebraic linear systems when the elements of $\tilde{A}$ and $\tilde{b}$ are LR fuzzy numbers. We use the theory of linear optimization to establish some new results for solvability of the systems. Next, we propose a linear programming problem to solve a fuzzy LR algebraic linear system.

As mentioned above, we here study fuzzy LR algebraic linear systems, $\tilde{A}x = \tilde{b}$, when the unknown vector, $x$, is a crisp (real) vector. Notice that in our previous works [19,21,20] (see also [36]), as discussed before, we studied fuzzy LR linear systems when the unknown vector is a fuzzy vector or equivalently $A\tilde{x} = \tilde{b}$. Therefore, the given results in this study on fuzzy LR algebraic linear systems are completely new and different from our given results in [19,21,20] on fuzzy LR linear system.

The necessary definitions and some properties of fuzzy arithmetic are provided in Section 2. We give some necessary and sufficient conditions for the solvability of a fuzzy LR algebraic linear system in Section 3. In Section 4, we introduce an approximate solution for a fuzzy LR algebraic linear system. In addition, we propose a linear programming problem to compute either a solution or an approximate solution. We conclude in Section 5.

## 2. Preliminaries

Here, we give some basic definitions and results concerning fuzzy numbers. A fuzzy number is a fuzzy quantity $A$ (see [37]) that represents a generalization of a real number $r$. Intuitively, $\mu_A(x)$ should be a measure of how well $\mu_A(x)$ "approximates" $r$, and certainly one reasonable requirement is that $\mu_A(r) = 1$ [37].

**Definition 2.1** [37]. A fuzzy number is a fuzzy quantity '$A$' that satisfies the following conditions:

1. $\mu_A(x) = 1$ for exactly one $x$.
2. The support $\{x : \mu_A(x) > 0\}$ of '$A$' is bounded.
3. The $\alpha$-cuts of '$A$' are closed intervals.

**Definition 2.2** [38]. The decreasing map $L : \mathbb{R}^+ \to [0,1]$ is called a shape function if the followings hold:

$$\begin{cases} L(0) = 1, \\ L(1) = 0, \\ 0 < L(x) < 1, \quad x \neq 0,1. \end{cases}$$

**Definition 2.3.** A fuzzy number $\tilde{A}$ is an LR-type if there exist shape functions $L$ (for left), $R$ (for right) and scalars $\alpha \geqslant 0, \beta \geqslant 0$ with

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right), & x \leqslant a, \\ R\left(\frac{x-a}{\beta}\right), & x \geqslant a. \end{cases}$$

The mean value of $\tilde{A}$, '$a$', is a real number, and $\alpha, \beta$ are called the left and right spreads, respectively. $\tilde{A}$ is denoted by $(a, \alpha, \beta)_{LR}$.

**Remark 2.4.** We can show a real number '$a$', by $(a, 0, 0)_{LR}$.

**Theorem 2.5** [38]. *Let* $\tilde{M} = (m, \alpha, \beta)_{LR}$, $\tilde{N} = (n, \gamma, \delta)_{LR}$ *and* $\lambda \in \mathbb{R}^+$. *Then,*

1. $\lambda\tilde{M} = (\lambda m, \lambda\alpha, \lambda\beta)_{LR}$.
2. $-\tilde{M} = (-m, \beta, \alpha)_{LR}$.
3. $\tilde{M} \bigoplus \tilde{N} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}$.