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Nonlinear Reynolds equation for hydrodynamic lubrication

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ABSTRACT

We derive a novel and rigorous correction to the classical Reynolds lubrication approximation for fluids with viscosity depending upon the pressure. Our analysis shows that the pressure dependence of viscosity leads to additional nonlinear terms related to the shear rate and arising from a non-negligible cross-film pressure. We present a comparison of the numerical solutions of the classical Reynolds equation and our modified equation and conclude that the modified equation leads to the prediction of higher pressures and viscosities in the flow domain.

1. Introduction

The Reynolds equation [1], which is an approximation of the classical Navier–Stokes equations, describes reasonably well the flow of a large class of fluids whose viscosities can be assumed to be independent of the pressure in many lubrication problems. However, there is a clear and incontestable evidence, that is well-documented, that attests to the fact that the viscosity varies with pressure, especially so in several problems concerning thin film lubrication, cf. [2–4]. Experiments have shown that in high pressure regimes pressure variations can significantly change the viscosity of certain lubricants and in areas such as elastohydrodynamic lubrication the classical isoviscous lubrication theory is noticeably incapable of explaining the existence of continuous lubricant films, for example, in rolling-contact bearings, cf. [4].

The traditional correction to the Reynolds approximation in the piezoviscous regime is based on a rather heuristic assumption that it suffices to replace the constant viscosity in the Reynolds equation by a suitable viscosity-pressure relationship, cf. [5]. This approach becomes questionable, however, in high pressure regimes because of the possible change of type (loss of ellipticity) of the equations and the potential existence of cross-film pressure gradient, see the discussion in [6]. In fact, for a Reynolds type approximation to be valid in this regime it ought to be derived from the full balance of linear momentum equations governing the flow of incompressible fluids with pressure-dependent viscosities. Now, the mathematical theory for the equations that govern the flows of fluids with a pressure-dependent viscosity has been advancing in leaps and bounds over the last few decades, see [7–15], but the only rigorous attempt to derive a modified Reynolds equation for lubrication problems in piezoviscous regime based upon these equations seems to have been carried out by Rajagopal and Szeri [6], see also [16] for further applications of their model.

In this paper we propose a new modified Reynolds equation for hydrodynamic lubrication in high pressure regimes. Our approach is built upon an asymptotic expansion of the non-dimensional velocity and pressure fields in terms of a small

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dimensionless parameter ϵ, related to the film thickness, and on a systematic analysis of the simplified set of equations obtained at different orders of ϵ, see [17] for a similar approach in the isoviscous case. Assuming that the dimensionless pressure–viscosity coefficient is of order ϵ, we show that the pressure distribution is governed by a Reynolds equation modified by a term depending, in particular, on the square of the shear rate. In [6], the authors assumed, for simplicity, that the cross-film pressure vanishes and derived a modified Reynolds equation, similar to ours, but depending on the elongation rate. Our computations show that it is exactly the (lower-order) cross-film pressure which is responsible for the new modified term in the Reynolds equation. We also prove that for smaller values of the pressure–viscosity coefficient the traditional modification of the Reynolds equation is a very accurate approximation of the pressure field.

It has been argued that the behavior of a piezoviscous fluid cannot be adequately described by a Reynolds type equation if the principal shear stress τ is not less than the reciprocal of the pressure–viscosity coefficient α, cf. [18]. This is not surprising since problems of nonexistence and nonuniqueness are expected for the full balance of linear momentum equations if $\tau \geqslant \alpha^{-1}$, cf. [7]. Similar conclusions can also be drawn from our modified Reynolds equations which ceases to be elliptic if the shear stress times the pressure–viscosity coefficient is larger than, or of the order of, one.

In the piezoviscous regime, the pressure-dependence of viscosity can be modeled through the Barus relation (Barus [19])

$$\mu = \mu_0 \, e^{\alpha p}, \tag{1}$$

where μ_0 denotes the dynamic viscosity measured at the ambient pressure ($p = 0$) and α is a positive parameter related to the rate of change of viscosity with respect to pressure, often referred to as the pressure–viscosity coefficient. The Barus relationship, together with the Roelands formula (Roelands [20])

$$\mu = \mu_0 \left(\frac{\mu_0}{\mu_R}\right)^{1-(1+p/p_R)^Z}, \tag{2}$$

where $\mu_R = 6.31 \cdot 10^{-5}$ Pa s, $p_R = 1.98 \cdot 10^8$ Pa and Z is a dimensionless parameter usually adjusted at ambient pressure to the Barus relation through the equation

$$Z = \frac{\alpha p_R}{\ln \mu_0 - \ln \mu_R} \tag{3}$$

are the most widely used models in elastohydrodynamic lubrication. For applications of these and other experimentally validated formulas for the variation of viscosity with pressure, temperature and density, see, e.g., [2,21–23,3,4].

In general, the pressure–viscosity coefficient α depends on the lubricant, and on the pressure, temperature and shear rate in the contact area, cf. [3]. For different lubricant oils, its value has been shown to vary between $1 \cdot 10^{-8}$ $(\mathrm{Pa})^{-1}$ and $4 \cdot 10^{-8}$ $(\mathrm{Pa})^{-1}$, cf. [24–26]. If the constant Z in the Roelands formula (2) is computed from Eq. (3), then both the Barus and the Roelands relation give $\frac{1}{\mu}\frac{d\mu}{dp} = \alpha$. We stress that although we rely on the assumption that the viscosity μ varies with pressure p according to the Barus or Roelands formula, our computations hold for more general pressure–viscosity relationships for which $\frac{1}{\mu}\frac{d\mu}{dp}$ is of the order of ϵ or smaller.

Before concluding this introductory section, it is worthwhile recognizing a marked departure of the constitutive relation of a piezoviscous fluid from that of the classical incompressible Navier–Stokes fluid with constant viscosity. While the latter is described by an explicit constitutive function for the stress in terms of the symmetric part of the velocity gradient, the former is an implicit relationship for the Cauchy stress and the symmetric part of the velocity gradient, leading to a totally different structure to the equations governing the flows of such fluids which in turn raises interesting issues with regard to both the mathematical and numerical analysis of the governing equations.

A one-dimensional rate type implicit model to describe the non-Newtonian response of fluids was introduced by Burgers [27]. His model includes as a special case the pioneering model to describe the viscoelastic response of fluids that was advanced by Maxwell [28]. Maxwell's model is however not an implicit model as the symmetric part of the velocity gradient can be expressed explicitly in terms of the stress and the time rate of the stress. Oldroyd [29] developed a systematic procedure to generate properly invariant three-dimensional rate type implicit constitutive relations. Such fluid models can be used to describe the flow of viscoelastic fluids and it is our aim to generalize the type of approximation that is being carried out here to include rate type implicit constitutive relations.

2. Lubrication approximation

Consider the following equations governing the isothermal flow of an incompressible, homogeneous, viscous fluid

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) - 2\,\nabla\cdot(\,\mu(p)\mathbf{D}(\mathbf{v})\,) + \nabla p = \rho\mathbf{b}, \tag{4}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{5}$$

where $\rho > 0$ is the constant density of the fluid, $\mathbf{v} = (u, v, w)$ is the velocity field and p is a scalar variable, often referred to as the mechanical pressure, associated with the incompressibility constraint (5). Moreover, we have assumed above that the