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A Total Lagrangian based method for recovering the un-deformed configuration in finite elasticity

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ABSTRACT

The problem of finding the un-deformed configuration of an elastic body, when the deformed configuration and the loads are known, occurs in many engineering applications. Standard solution methods for such problems include conservation laws based on Eshelby's energy–momentum tensor and re-parameterization of the standard equilibrium equations. In this paper we present a different method for solving such problems, based on a re-parameterization of the nodal forces using the Total Lagrangian formulation. The obtained nonlinear system of equations describing equilibrium can be solved using either Newton–Raphson or an explicit dynamic relaxation algorithm. The solution method requires only minor modifications to similar algorithms designed for forward motion calculations. Several examples involving large deformations and different boundary conditions and loads are presented.

1. Introduction

The problem of finding the un-deformed shape of an elastic body, when the final deformed shape, material behavior, applied loads and boundary conditions are known, is encountered in many different engineering applications. A classic example is the design of rubber seals, which are pressed into a channel and need to exert a desired pressure onto the channel. Another application is the design of rubber forms to be used in pressing thin sheets of metal in stamping procedures [1].

This problem stated above has been called by some authors an "inverse elastostatics" problem [2–4]. As many elasticity problems related to model parameter identification are also called inverse problems, in order to avoid confusion we will use the term "direct deformation" when referring to the computation of deformation in an elastic body under known loads starting from the un-deformed configuration and the term "inverse deformation" when referring to the computation of deformations in an elastic body under known loads starting from the deformed configuration.

More recently the need for solutions to inverse deformation problems was also identified in various areas of bio-engineering. In [5] an application to breast biomechanics is presented. The imaging of the breast is performed with the patient lying in a prone position (on the stomach). Surgical procedures such as breast biopsy are usually performed with the patients lying on their back (supine position). The reference un-deformed state of the breast is needed in order to predict the breast shape and perform image registration to the new patient position. In [2,3,6,7] the solution of an inverse deformation problem is used in order to assess stresses in the walls of aneurysms. Sometimes the solution to an inverse deformation problem is avoided by

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modeling the structure in such a way that the stresses can be computed directly based on the geometry and applied loads. For example, aneurysms are modeled as membranes in [8].

A solution method for inverse deformation problems in finite elasticity was proposed in [9], by formulating the problem as a set of balance equations written in terms of inverse deformation and standard boundary conditions. This formulation exploits a set of duality relations that allow the formulation of an inverse deformation problem in a form that appears similar to a standard elastostatic problem. Later, Chadwick [10] re-formulated the equilibrium equations in terms of Eshelby's energy–momentum tensor [11].

In [4] Govindjee and Mihalic show that the energy–momentum formulation has several deficiencies: it places strong continuity requirements on the motion, Eshelby's tensor lacks direct physical connection to the stated problem creating difficulties with the boundary conditions and it cannot handle body forces. They propose a new method based on the re-parameterization of the equilibrium equations, which eliminates these difficulties. The method is later extended to incompressible materials in [1], and was shown to be consistent with the approach presented in [12].

The methods presented so far require the writing of the stress equilibrium equations in terms of the deformed configuration (Eulerian description). Conventional forward finite elasticity analyses are typically formulated with respect to a Lagrangian frame of reference (based on the un-deformed configuration) [13]. Rajagopal et al. present a method of solving inverse deformation problems using a Lagrangian frame of reference in [5]. Their method is based on a finite difference approximation of the Jacobian for the system of equations describing the equilibrium, considering the parameters of the deformed state as an initial estimate for the parameters of the reference state. In [7] Riveros et al. propose a method which uses the displacements obtained by solving the direct problem using the current configuration to iteratively update the geometry, which should converge towards the un-deformed geometry.

In this paper we present a new method of solving inverse deformation problems. The starting point for our method consists of the standard equilibrium equations discretised using a Total Lagrangian (TL) framework. The direct elastostatic problem can be stated, after discretisation, as an equation defining the equilibrium of forces with the displacements as unknowns. The TL framework allows a direct relationship between the un-deformed and the deformed configurations to be defined. We exploit this relationship to rewrite the force equilibrium equation based on the known deformed configuration. The obtained non-linear system of equations can be solved using Newton–Raphson techniques, and we present the relations needed for the construction of the exact stiffness matrix required. We also present a different solution method based on explicit time integration [14] and dynamic relaxation [15].

The paper is organized as follows: the derivation of the proposed method is presented in Section 2, several examples involving different boundary conditions and loads are presented in Section 3, followed by discussion and conclusions in Section 4.

2. Problem formulation and solution method

2.1. Derivation of equilibrium equation

We consider hyperelastic materials for which the internal stresses and strains depend only on the deformation field within the material and are independent of the way the deformation was applied (path-independent). For such materials the constitutive behavior is usually defined using a strain energy potential function:

$$ {}_{0}^{t}W = f(I_1, I_2, J, \ldots), \tag{1} $$

where I_1 and I_2 are the first and second strain invariants, computed based on the stretch matrix (Cauchy–Green strain tensor) and J is the total volume change, computed as the determinant of the deformation gradient [13]. The left subscript identifies the starting configuration (in this case the un-deformed state, 0) and the left superscript identifies the current configuration. For anisotropic materials there may be other parameters describing the anisotropic behavior (such as the direction of fibers). Our derivation is based on the isotropic case, but can be easily extended to handle anisotropy.

Based on the strain energy potential the second Piola–Kirchhoff stress, $\mathbf{S}$, can be computed as:

$$ {}_{0}^{t}\mathbf{S}({}_{0}^{t}\mathbf{F}) = \frac{\partial {}_{0}^{t}W}{\partial {}_{0}^{t}\mathbf{E}}, \tag{2} $$

where $\mathbf{E}$ is the Green–Lagrange strain. Because both the strain energy potential and the Green–Lagrange strain are defined based on the deformation gradient $\mathbf{F}$, the second Piola–Kirchhoff stress can be considered as a function of the deformation gradient. The deformed configuration of an object can be characterized using the equilibrium between the internal forces in the final deformed state f and the externally applied forces:

$$ {}_{0}^{f}\mathbf{R}_{\mathbf{int}} = {}_{f}\mathbf{R}_{\mathbf{ext}}. \tag{3} $$

After the weak form of the equilibrium equations is discretised (for example using a finite element discretisation), the deformation field can be described using the nodes of the discretisation and their associated shape functions. The internal forces can be expressed, after discretisation using the Total Lagrangian formulation, as [16,17]: