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# A *a posteriori* regularization for the Cauchy problem for the Helmholtz equation with inhomogeneous Neumann data ☆

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ABSTRACT

In this paper the Cauchy problem for the Helmholtz equation with inhomogeneous Neumann data is considered. This problem is severely ill-posed, the solution does not depend continuously on the data. An approximate method based on the *a posteriori* Fourier regularization in the frequency space is analyzed. Some crucial information about the regularization parameter hidden in the *a posteriori* choice rule are found, and some sharp error estimates between the exact solution and its regularization approximate solution are proved. Numerical examples show the effectiveness of the method. A comparison of numerical effect between the *a posteriori* and the *a priori* Fourier method is also taken into account.


## 1. Introduction

In this paper we specifically consider a Cauchy problem for the Helmholtz equation with only inhomogeneous Neumann data.

The Helmholtz equation arises naturally in many physical applications, in particular related to acoustic or electromagnetic wave propagation.

The Cauchy problem for the Helmholtz equation with a real wave number $k$ on a "strip" domain is as follows:

$$\begin{cases} \Delta u(x,y) + k^2 u(x,y) = 0, & x \in (0,1),\ y \in \mathbb{R}^n, \\ u(0,y) = \varphi(y), & y \in \mathbb{R}^n, \\ u_x(0,y) = \psi(y), & y \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}$ is a $(n+1)$-dimensional Laplace operator, $\varphi(y),\ \psi(y) \in L^2(\mathbb{R}^n)$ are the Dirichlet and Neumann data, respectively. Some reasons for investigating this problem following from optoelectronics and in particular in laser beam models have been explained in detail in [1].

The Cauchy problem for the Helmholtz equation is an inverse problem [2] and is severely ill-posed, i.e., the solution does not depend continuously on the data, any small change of the data may cause dramatically large error in the solution. Therefore, some regularization methods for solving this problem is important and necessary.

Due to the linearity of problem (1.1), it can be divided into two problems with only one inhomogeneous Dirichlet or Neumann data, respectively:

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$$\begin{cases} \Delta v(x,y) + k^2 v(x,y) = 0, & x \in (0,1),\ y \in \mathbb{R}^n, \\ v(0,y) = \varphi(y), & y \in \mathbb{R}^n, \\ v_x(0,y) = 0, & y \in \mathbb{R}^n \end{cases} \tag{1.2}$$

and

$$\begin{cases} \Delta w(x,y) + k^2 w(x,y) = 0, & x \in (0,1),\ y \in \mathbb{R}^n, \\ w(0,y) = 0, & y \in \mathbb{R}^n, \\ w_x(0,y) = \psi(y), & y \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

It is obvious that if $v(x,y)$ and $w(x,y)$ are the solutions of problem (1.2) and (1.3), respectively, then $u(x,y) = v(x,y) + w(x,y)$ is the solution of problem (1.1).

For problem (1.2), there have been many results and various methods have also been presented. A *a priori* Fourier regularization method was firstly applied in [1], subsequently, this method has been further improved in [3]. Under the $L^2(\mathbb{R}^n)$ *a priori* bound assumption, the optimal error bound for approximate solution was derived and also the spectral and a revised Tikhonov regularization methods were presented in [4]. The conditional stability estimates in the general source condition was used to the optimality analysis of the approximate solution in [5]. Two quasi-reversibility methods were presented in [6,7], respectively. A modified Tikhonov method was presented in [8]. A quasi-boundary-value method was presented in [9]. As an application of the general theory of the *a priori* Fourier method, Example 3.3 in [10] also considered this problem. The regularization methods used in the vast majority of works on problem (1.2) are *a priori* methods, only [5,11] involve *a posteriori* regularization methods.

However, to the authors' knowledge, there are few works devoted to the error estimates of regularization methods for problem (1.3), in which the *a priori* Fourier method was applied in [10], and both the *a priori* and the *a posteriori* modified Tikhonov method was applied in [12].

In this paper we will apply the *a posteriori* Fourier method to solve problem (1.3). The reasons are listed below:

1. For any linear ill-posed problems defined on a "strip" domain, the Fourier method, which was first applied to the inverse heat conduction problem by Eldén et al. in [13], is the most simple and a very effective regularization method.
2. The general theory under the frame of numerical pseudodifferential operators for the *a priori* and the *a posteriori* Fourier method was preliminarily established in [10,11], respectively. It's a little pity that the *a posteriori* theory given in [11] is unable to cover the Cauchy problem for the Laplace equation with inhomogeneous Neumann data and more is not applicable to problem (1.3). For the former, it has been considered in [14], but relatively, the research of problem (1.3) has more particular difficulty and also has independent significance for perfecting the theory of the *a posteriori* Fourier method.
3. For any the *a priori* regularization method, the choice of the regularization parameter usually depends on both the *a priori* bound and the noise level. In general, the *a priori* bound cannot be known exactly in practice, and working with a wrong *a priori* bound may lead to bad regularization solution. The advantage of the *a posteriori* method is that one does not need to know the smoothness and the *a priori* bound of unknown solution. So it is particularly worthy of further development. However, because some important information about the solution are concealed and hidden for the Morozov's discrepancy principle, such that the theoretical analysis of the convergence rate with high-accuracy of the approximate solution is rather difficult. For example, the existing works for problem (1.3) about the *a posteriori* method only involve weaker $L^2$-smoothness assumption [5,12]. Therefore, some new ideas are needed to obtain more profound and deeper results for problem (1.3).
4. For problem (1.1)–(1.3), some rather sharp restriction for the wave number $k$ was imposed in some known important works. For example, the condition $dk < \frac{\pi}{2}$ was required in [1,5], where $d$ is the width of the "strip" domain (without loss of generality, we take $d = 1$ in this paper, i.e., $x \in (0,1)$). Such a restriction will bring inconvenience to practical application, and some improvements are needed.

The main aim of this paper is to solve the above puzzles effectively using the *a posteriori* Fourier method for problem (1.3). This paper is organized as follows. In Section 2, a new non-standard *a priori* bound for unknown solution is introduced, which is a refinement of the standard $H^p(\mathbb{R}^n)$-*a priori* bound and can more accurately portray the degree of ill-posedness for problem (1.3) in the exponential sense [15,16,10,11]. Meanwhile, a conditional stability result is also given, which will ensure the theoretical stringency of the *a posteriori* rule in the next section. In Section 3, an *a posteriori* choice rule of regularization parameter based on the Morozov's discrepancy principle is given, and some deep-seated nature of the regularization parameter is stated, but the proof of main result is given in the appendix at the end of this paper. In Section 4, some convergence rate estimates with Hölder type or asymptotic logarithm-Hölder type are given. In Section 5, some numerical examples are provided, which show the effectiveness of the method. A comparison of numerical effect between the *a posteriori* and the *a priori* Fourier method is also taken into account in Example 5.1. This paper ends with a brief conclusion in Section 6.