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Explicitly uncoupled variational multiscale for characteristic finite element methods based on the unsteady Navier–Stokes equations with high Reynolds number

Gang Chen, Minfu Feng *

School of Mathematics, Sichuan University, Chengdu 610064, China

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ABSTRACT

In this study, we consider the combination of a known method for inducing variational multiscale treatments by postprocessing with known characteristic time stepping methods. Both are interesting for practical computational fluid dynamics applications but this combination has not been explored previously. We prove that the error estimates depend on a reduced Reynolds number. Numerical experiments show that our method improves the numerical performance of the straightforward characteristic method.

1. Introduction

Variational multiscale (VMS) methods are efficient and simple approaches for the numerical simulation of turbulent flows. They were first proposed by Hughes in [1]. The basic idea of VMS methods is to define large scales by projections into appropriate function spaces. Many previous studies [2–10] have considered VMS methods.

The success of VMS methods leads naturally to the question of how to introduce them into existing legacy codes. In [11,12], the authors proposed the application of separate, uncoupled, and modular postprocessing steps each time during the flow code in VMS methods. An uncoupled postprocessing step can be implemented in legacy codes to recover the VMS eddy viscosity term. In previous studies, the legacy codes were assumed to employ the nonlinear Crank–Nicolson method for the Navier–Stokes equations. Numerical experiments have demonstrated the efficiency of these postprocessing methods. However, a major problem is that the error estimates in these methods depend totally on the Reynolds number. Thus, the error estimates in these methods are useless for high Reynolds number problems, so better error estimates are needed. Another problem is whether this postprocessing method is suitable for legacy codes other than nonlinear Crank–Nicolson schemes, such as the characteristic method for the Navier–Stokes equations. These two issues motivated the present study.

The characteristic methods were proposed for the numerical treatment of convection-dominated diffusion equations. In these methods, the governing equations are rewritten in terms of Lagrangian coordinates defined by the particle trajectories (or characteristics) associated with the problem under consideration. This Lagrangian treatment greatly reduces the time

* Corresponding author. Tel.: +86 13908174035.
E-mail address: fmf@wtjs.cn (M. Feng).

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truncation error. The characteristic methods are used to solve the NSEs [13–15]. However, the convective effects still need to be stabilized. Thus, we propose a postprocessing method that improves the accuracy of the original characteristic methods.

In this study, we propose the combination of a known method for inducing VMS treatments by postprocessing with known characteristic time stepping methods. Both are useful for practical computational fluid dynamics (CFD) applications but their combination has not been explored previously. This method includes a VMS treatment of turbulence in the legacy code of the characteristic methods. We prove that the error estimates depend on a reduced Reynolds number. The numerical experiments reported in Section 5.1 verify the convergence of our method, thereby demonstrating that our method improves the numerical performance of the straightforward characteristic method. In Section 5.2, we compare our method with VMS methods [5] based on flow through a three-dimensional channel. The results shows that the performance of our method is comparable with that of VMS methods.

The remainder of this paper is organized as follows. In Section 2, we introduce the requisite notations. In Section 3, we describe our proposed postprocessing VMS method for the characteristic method. In Section 4, we present an analysis of the stability and error estimates. In Section 5, we present the results of numerical experiments, which illustrate and confirm our theoretical analysis. In Section 6, we give the conclusions of our study.

Throughout this paper, we use C to denote a positive constant that is independent of $\Delta t, h$, and ν, which is not necessarily the same at each occurrence.

2. Basic notations

Let $\Omega \in \mathbb{R}^d (d = 2, 3)$ be a bounded domain with a polygonal or polyhedral boundary $\Gamma = \partial\Omega$. We use $W^{m,p}(\Omega), W_0^{m,p}(\Omega)$ to denote the m-order Sobolev space on Ω, and we use $\|\cdot\|, |\cdot|$ to denote the norm and semi-norm on these spaces. When $p = 2$, we let $H_0^m(\Omega) = W_0^{m,p}(\Omega), H^m(\Omega) = W^{m,p}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,p}, |\cdot|_m = |\cdot|_{m,p}$, the inner product of $H^m(\Omega)$ is denoted by $(\cdot,\cdot)_m$. When $m = 0$, we let $(\cdot,\cdot) = (\cdot,\cdot)_m$. The space $L_0^2(\Omega)$ denotes the space $\{v \in L^2(\Omega) : \int_\Omega v dx = 0\}$. Let X denote a Banach space, with the mapping $\phi(x,t) : [0,T] \to X$, and we define

$$\|\phi\|_{L^2(0,T;X)} = \left(\int_0^T \|\phi\|_X^2(t)dt\right)^{1/2}, \quad \|\phi\|_\infty = \sup_{0\leqslant t\leqslant T} \|\phi\|_X(t). \tag{2.1}$$

Vector analogues of the Sobolev spaces and the vector-valued functions are denoted by upper and lower case bold face fonts, respectively, e.g., $\boldsymbol{H}_0^1(\Omega), \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{u}$.

Let $I = [0,T]$, where T is a positive constant. We consider the unsteady Navier–Stokes equations,

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \times I, \\ \nabla\cdot\boldsymbol{u} = 0 \in \Omega\times I, \\ \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma\times I, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \text{ in } \Omega, \end{cases} \tag{2.2}$$

where $\boldsymbol{u} = (\boldsymbol{x},t) \in \mathbb{R}^d$ denote the velocities, $p = p(\boldsymbol{x},t) \in \mathbb{R}$ denotes the pressure, $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t) \in \mathbb{R}^d$ denote the body forces, $\nu = Re^{-1}$ denotes the viscosity coefficient, and Re denotes the Reynolds number. Let $\boldsymbol{V} = \boldsymbol{H}_0^1(\Omega), Q = L_0^2(\Omega)$ and $\boldsymbol{W} = \{\boldsymbol{v} \in \boldsymbol{V} | \nabla\cdot\boldsymbol{v} = 0\}$.

Following the standard procedure in [13], we define the derivative of $\boldsymbol{u}$ in the direction of flow $\boldsymbol{u}$. Let $\psi = (1 + |\boldsymbol{u}|^2)^{1/2}$, and $\alpha_t, \alpha_1, \ldots, \alpha_d$ be real numbers from the interval $(0, 2\pi]$ such that $\cos\alpha_t = \frac{1}{\psi}, \cos\alpha_i = \frac{u_i}{\psi}, i = 1,\ldots,d$. Then, the characteristic direction of $\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is defined as,

$$\begin{aligned} &\frac{\partial}{\partial\tau} = \cos\alpha_t\frac{\partial}{\partial t} + \sum_{i=1}^d \cos\alpha_i\frac{\partial}{\partial x_i}, \\ &D_t\boldsymbol{u} = \boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \psi\frac{\partial\boldsymbol{u}}{\partial\tau}. \end{aligned} \tag{2.3}$$

Let $\nabla t = T/N$ be the time step, where N is a positive integer and $t_n = n\Delta t$. We use the expression

$$g^n = g(\boldsymbol{x}, t_n). \tag{2.4}$$

Then, a weak formulation of problem (2.2) at time level $n+1$ is

$$(D_t\boldsymbol{u}^{n+1}, \boldsymbol{v}) + B(\boldsymbol{u}^{n+1}, p; \boldsymbol{v}, q) = (\boldsymbol{f}^{n+1}, \boldsymbol{v}), \tag{2.5}$$

where $B(\boldsymbol{u},p;\boldsymbol{v},q) = \nu(\nabla\boldsymbol{u},\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v},p) + (\nabla\cdot\boldsymbol{v},q)$. To discretize the term $D_t\boldsymbol{u}$, we denote $\boldsymbol{X}(\boldsymbol{x}, t_{n+1}; t)$ as the characteristic curves associated with the material derivative, which is defined by the following initial value problem

$$\begin{cases} \frac{d\boldsymbol{X}(\boldsymbol{x},t_{n+1};t)}{dt} = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{x},t_{n+1};t),t), \\ \boldsymbol{X}(\boldsymbol{x},t_{n+1};t_{n+1}) = \boldsymbol{x}. \end{cases} \tag{2.6}$$