Applied Mathematical Modelling 40 (2016) 19–30

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Exponential stability of the exact and numerical solutions for neutral stochastic delay differential equations

Xiaofeng Zong [a,*], Fuke Wu [b]

[a] Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China
[b] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, People's Republic of China

ARTICLE INFO

Article history:
Received 22 December 2012
Received in revised form 20 March 2015
Accepted 14 May 2015
Available online 21 May 2015

Keywords:
Neutral stochastic delay differential equations
Moment exponential stability
Euler Maruyama method
Backward Euler Maruyama method

ABSTRACT

This paper is concerned with pth moment and almost sure exponential stability of the exact and numerical solutions of neutral stochastic delay differential equations (NSDDEs). Moment exponential stability criteria of the continuous and discrete solutions are established by virtue of the Lyapunov method. Then the almost sure exponential stability criterion is derived by the Chebyshev inequality and the Borel–Cantelli lemma. We also examine conditions under which the numerical solution can reproduce the exponential stability of exact solution. It is shown that the linear growth condition is necessary for Euler–Maruyama (EM) method to maintain the moment exponential stability of the exact solution. If the drift coefficient of NSDDE satisfies the one-sided Lipschitz condition, EM method may break down, but we show that the backward EM (BEM) method can share the mean square exponential stability of the exact solution.

1. Introduction

Stochastic differential systems, including stochastic differential equations (SDEs), stochastic delay differential equations (SDDEs) and NSDDEs, have been greatly developed and play an important role in many ways such as economics, finance, physics, biology, medicine, and other science. Recently, stability theorems of stochastic differential systems, for example, moment stability (M-stability, see [1,2]) and almost sure stability (or the trajectory stability (T-stability), see [3]), have attracted much attention. There are many results on stability theorems for stochastic differential systems (see Mao's book [4,5]). Some of the stability criteria related to the pth moment exponential stability of the solutions to neutral stochastic functional differential equations (NSFDEs) were considered in [6–14] and the references therein.

Due to that most stochastic differential systems cannot be solved explicitly, numerical methods have become essential. The stability theory of numerical solutions is one of fundamental research topics in the numerical analysis. The stability of numerical solutions for SDEs has received increasing attention in recent years (see [2,3,15–19] and the related references therein). The continuous and discrete semi-martingale convergence theorems are important tools for investigating the almost surely asymptotic stability of the continuous and discrete stochastic systems (see [17,19–21]). The advantage of discrete semi-martingale convergence theorem is that it produces almost sure exponential stability of the numerical solution directly without resorting to the Chebyshev inequality and the Borel–Cantelli lemma (see [20,21]). Many numerical results for stochastic delay differential equations of neutral type focused on convergence and asymptotic stability (see [22–28]). To

* Corresponding author.
E-mail addresses: xfzong87816@gmail.com (X. Zong), wufuke@mail.hust.edu.cn (F. Wu).

http://dx.doi.org/10.1016/j.apm.2015.05.001
0307-904X/

date, for NSDDEs, the continuous and discrete semimartingale convergence theorems succeeded in obtaining almost surely asymptotic stability of the exact and numerical solutions (see [25]). So far little is known about the exponential stability and the decay rate of the numerical solutions to this NSDDEs and the aim of this paper is to close this gap.

This paper devotes to investigate the moment exponential stability of the exact and numerical solutions of NSDDEs and examine conditions under which the numerical solutions of NSDDEs can preserve the mean square stability of the exact solution. Then the almost sure exponential stability can be obtained by the Chebyshev inequality and the Borel–Cantelli lemma. The new stability criterion of the exact solution, we will establish, gives a more accurate bound of Lyapunov exponent than the existed results. Stability analysis aims to study whether the numerical methods can recover the moment exponential stability of the exact solution for the small timesteps. It is shown that the linear growth condition is necessary for the EM method to reproduce the moment exponential stability. A counterexample will be introduced to show that EM method may not be able to reproduce this stability of the exact solution if the drift coefficient f does not satisfy the linear growth condition. However, the BEM method produces good results under a one-sided Lipschitz condition, which is a less restrictive condition than the linear growth condition.

The rest of the paper is arranged as follows. The next section provides necessary notations and states the definitions on pth moment exponential stability and almost sure exponential stability for the use of this paper. In Section 3, the criteria related to the pth moment exponential stability of the exact and numerical solutions to NSDDEs are considered. Section 4 presents some conditions under which EM and BEM approximations may reproduce the exponential mean square stability of the exact solution. Section 5 gives the conclusion to end the work.

2. Notations and definitions

Throughout this paper, unless otherwise specified, we use the following notations. Let $|\cdot|$ be the Eucilidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathfrak{F}_0$ contains all P-null sets. $w(t)$ is a d-dimensional Brownian motion on this probability space. Let $\mathbb{R}_+ = [0,\infty)$ and $\tau > 0$. Denoted by $C([-\tau,0],\mathbb{R}^n)$ the family of continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leqslant\theta\leq 0}|\varphi(\theta)|$. Let $C^b_{\mathfrak{F}_0}([-\tau,0],\mathbb{R}^n)$ be the family of all $\mathfrak{F}_0$-measurable bounded $C([-\tau,0],\mathbb{R}^n)$ valued random variables $\xi = \{\xi(\theta) : -\tau \leqslant \theta \leq 0\}$. The inner product of $X, Y \in \mathbb{R}^n$ is denoted by $\langle X, Y\rangle$ or X^TY. We use $a \vee b$ to denote $\max\{a,b\}$ and $a \wedge b$ to denote $\min\{a,b\}$.

We consider the following n-dimensional nonlinear NSDDE

$$d(x(t) - u(x(t-\tau))) = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dw(t), \quad t \geqslant 0, \tag{2.1}$$

with initial data $x_0 = \xi$, where $\xi = \{\xi(\theta) : -\tau \leqslant \theta \leq 0\}, u : \mathbb{R}^n \to \mathbb{R}^n, f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times d}$, For the purpose of stability, we assume that $u(0) = 0$ and $f(0,0,t) = g(0,0,t) = 0$. That means (2.1) admits a trivial solution. Then we give some assumptions for u, f and g, under which there exists a unique local solution $x(t) = x(t;\xi)$ to Eq. (2.1).

Assumption 2.1 (Local Lipschitz condition). f and g satisfy the local lipschitz condition, that is, for each $j > 0$ there exists a positive constant C_j such that

$$|f(x,y,t) - f(\bar{x},\bar{y},t)| \vee |g(x,y,t) - g(\bar{x},\bar{y},t)| \leqslant C_j(|x-\bar{x}| + |y-\bar{y}|) \tag{2.2}$$

for all $t \geqslant 0$ and $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leqslant j$.

Assumption 2.2 (Contractive mapping). u is a contractive mapping, that is, there exists a positive constant $\kappa \in (0,1)$ such that

$$|u(x) - u(y)| \leqslant \kappa|x-y| \tag{2.3}$$

for all $x, y \in \mathbb{R}^n$.

Let $C^2(\mathbb{R}^n;\mathbb{R}_+)$ denote the family of all nonnegative functions $V(x)$ on $\mathbb{R}^n$ which are continuously twice differentiable in x. For each $V(x) \in C^2(\mathbb{R}^n;\mathbb{R}_+)$, define an operator LV from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}$:

$$LV(x,y,t) = V_x(x-u(y))f(x,y,t) + \frac{1}{2}\mathrm{trace}\left[g^T(x,y,t)V_{xx}(x-u(y))g(x,y,t)\right]. \tag{2.4}$$

To facilitate reading, we give the definitions on the pth moment exponential stability and almost sure exponential stability of the exact and numerical solutions (see [4,29]).

Definition 2.1. The trivial solution $x(t)$ to Eq. (2.1) is said to be pth moment exponentially stable (or almost surely exponentially stable) if there exists a constant $\eta > 0$ such that