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Forced flow and solidification over a moving substrate

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ABSTRACT

We investigate fluid flow and solidification of a binary alloy on a moving substrate. We derive asymptotic solutions for the flow, thermal and concentration fields as well as the growth rate of the solid–liquid interface in the limit of small Prandtl number. We quantify the effect of a forced flow in the melt on the growth characteristics of a solid–liquid interface and the boundary-layer structure at the interface. We also examine the influence of the forced flow on a local advective flux of solute in the melt.

1. Introduction

The solidification of fluids is an integral part of many natural and industrial processes. Among typical examples is the formation of snowflakes and icicles in winter or the sea ice in polar areas. The phase-change processes play an important role also in material engineering during production of new materials such as metal castings and semiconductors.

It is a well known phenomenon that a material solidifying from an alloy has usually different composition than that of the original system. For example, the ice growing from sea water is almost pure. The way in which the liquid material solidifies can affect the quality of the final solidified product. A typical example is the appearance of structural defects, called freckles, during solidification of metal alloys (see [1]). To control the quality of solidified products, it is necessary to understand the coupling between fluid flow and solidification involved.

A mathematical model for diffusion-driven solidification of a binary alloy cooled below was studied in [2] as an extension of the classical Stefan problem for a single component system. The interface between the solid and liquid phases, characterised by the local conservation of heat and solute, was assumed planar. The rate of solidification in the model was controlled by the diffusive transport of solute away from the interface. Analytical self-similar solutions were found, with square-root time growth of the interface. The evolution of a perturbed, initially planar solid–liquid interface during the diffusion-driven solidification of a binary alloy was studied in [3].

Of different nature is a so-called directional solidification in which the solidifying liquid is pulled at constant speed through a constant temperature gradient and the solid–liquid interface is stationary. In recent years, the models of directional solidification were studied extensively in context of morphological and convective instabilities (see [4,5]).

An experimental configuration that is common in material engineering is the one in which a cooled horizontal boundary (substrate) is moving at a constant speed in horizontal direction in an imposed vertical temperature gradient (continuous strip and spin casting). There are two main features that distinguish such a configuration from those with a stationary cooled

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boundary or those of directional solidification: (i) the solidifying interface is not planar; (ii) there is a strong two-dimensional flow in the liquid phase. Such a configuration was previously addressed as a local approximation of spin casting (see the review article [6]). The initial solidification of a pure liquid–metal film flow over a moving boundary was studied in [7]. The problem of a steady two-dimensional boundary layer flow of a binary alloy over a moving substrate was studied in [8], where self-similar solutions for the velocity, temperature and solute concentration fields were found in the limit of small Prandtl number, which is typical of liquid metal flows. The interface was shown to have a square-root growth in the horizontal direction. The self-similar analysis was facilitated by the assumption of semi-infinite domain in vertical direction and that of small interfacial slope. An extension of the problem to include a two-phase, or mushy, region was considered in [9,10]. The mushy layer consisted of two separate layers: a packing region, with solid phase moving with the substrate, and a dispersed region, where solid phase was free to move with the fluid. The self-similar solutions were found numerically.

In this paper, we investigate a problem originally formulated in [8], but employing a different scaling of the governing equations. As in [8], we assume the asymptotic limit of small Prandtl number. This limit is singular for the flow field, corresponding to a thin viscous boundary layer adjacent to the solid–liquid interface. For the temperature and solute concentration field, however, this limit is regular, which simplifies the asymptotic analysis. From the experimental point of view, of central importance is the ratio of the horizontal flow velocity forced at the infinity to the pulling rate of the substrate. It is through this velocity ratio that the flow controls the solidification. The main motivation of the present paper is the assessment of how this velocity ratio affects the solidification, particularly the asymptotic limit when this ratio is small.

The organisation of the paper is as follows. In Section 2 we describe the physical model and formulate the dimensional equations governing the transport of heat, solute and momentum in the liquid region and of heat in the solid region. We then nondimensionalise the problem and, in order to facilitate the self-similar analysis, make a boundary-layer reduction in the limit of small interfacial slope. In Section 3 we present the analytical results for the flow, thermal and concentration fields in the limit of small Prandtl number, together with an eigenvalue relation for the dimensionless growth rate. In Section 4 we derive the approximate forms of solutions in the limit of small velocity ratio. A closed form for advective flux of solute is also presented. Finally, in Section 5 we provide some numerical estimates for a real physical system and discuss the results obtained.

2. Mathematical formulation

We consider a binary alloy occupying the region $x > 0$ and $z > 0$, which solidifies over a cooled plate $z = 0$ moving horizontally at constant speed $U_0 > 0$. The temperature of the plate is maintained at a value T_0 that is assumed to be above the temperature corresponding to the eutectic point of the binary phase diagram (below which the system is completely solidified) and below the liquidus temperature $T_L(C_\infty)$ corresponding to the far-field solute concentration C_∞ at $z \to \infty$. The far-field temperature in liquid is T_∞. A definition sketch for the problem under consideration is depicted in Fig. 1. A typical phase diagram for solidifying binary alloy is shown in Fig. 2.

The interface is assumed to be in local thermodynamic equilibrium, so that its temperature T_h and the composition on the liquid side of the interface, C_h, are related by the liquidus relationship

$$T_h = T_L(C_h) \equiv T_0 - \hat{\Gamma}(C_h - C_0), \tag{2.1}$$

where $\hat{\Gamma} > 0$ is the liquidus slope and C_0 is such that $T_0 = T_L(C_0)$. We assume complete solute rejection so that the solid is free from solute. We also assume that there is no mass diffusion in the solid.

We denote by $\mathrm{D}h/\mathrm{D}t = -\boldsymbol{u}_0 \cdot \boldsymbol{n}$ the local velocity of the solid material elements relative to the (stationary) solid–liquid interface, where $\boldsymbol{u}_0 = U_0\boldsymbol{i}$ is the velocity of the solid material elements relative to the stationary frame of reference, $\boldsymbol{i}$ being the unit vector in the horizontal direction and $\boldsymbol{n}$ is the outward unit vector normal to the interface. Then, the dimensional heat and solute conservation laws at the solid–liquid interface can be formulated as

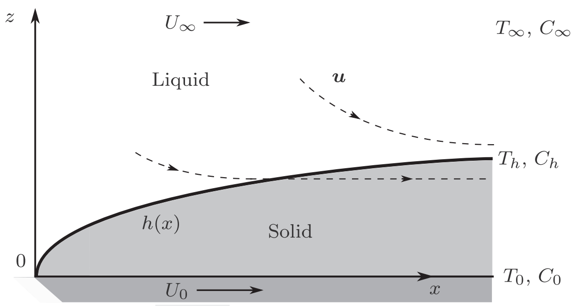

Fig. 1. A definition sketch for the problem of solidification of a binary alloy over a horizontally moving substrate. A semi-infinite region $x > 0, z > 0$ is filled with a binary alloy of far-field solute concentration C_∞ and temperature T_∞ (for $z \to \infty$ and x fixed). The cooled lower boundary lies in the plane $z = 0$ and is moving in horizontal direction at a constant speed U_0. The stationary solid–liquid interface is located at $z = h(x)$. The horizontal flow velocity at $z \to \infty$ is U_∞.