Applied Mathematical Modelling 40 (2016) 41–49

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

A model for pattern formation under gravity

Mayer Humi

Department of Mathematical Sciences, Worcester Polytechnic Institute, 100 Institute Road, Worcester, MA 01609, United States

ARTICLE INFO

Article history:
Received 12 May 2014
Received in revised form 30 December 2014
Accepted 17 April 2015
Available online 21 May 2015

Keywords:
Self gravitating gas
Ring formation
Hydrodynamic model

ABSTRACT

In this paper we present an analytic hydrodynamic model for the steady state formation of a ring pattern in the density distribution by a self gravitating, incompressible and stratified disk of gas. We consider also the possible astrophysical applications of this model.

1. Introduction

Recent astronomical discoveries lead us to believe that our solar system is not unique. In fact the reverse is true viz. a large number of stars have planets orbiting around them and the number of known exoplanets at the present time is around two thousands [1–3]. This data leads to the hypothesis that there is a fundamental physical process which we do not understand fully as yet that leads to the formation of planetary systems throughout the galaxy (and beyond).

Many theories were put forward in the past about the origin of the solar system. These include Jeans tidal theory, Alfven band structure model and many others [4–6]. (See also [7–10] for a complete list of references.) Originally it was Laplace [11] in 1796 who put forward the hypothesis that planetary systems evolve from a family of isolated rings that were formed from a primitive interstellar gas cloud. More recently Berlage [12] and Prentice [13] followed on Laplace hypothesis and explored possible mechanisms for the formation of ring structure in the solar nebula. (Actually such a system of rings around a protostar was observed recently by the Atacama Large Millimeter/submillimeter Array in the constellation Taurus).

Another model for planetary formation was the vortex model put forward by Von Weizsacker [14] and elaborated further by Chandrasekhar [15]. Von Weizsacker proposed the existence of a system of vortices in the annular regions of the rotating solar nebula. According to this model proto-planet formation would occur at the boundaries of adjacent counter rotating vortices.

Currently the leading theory about the formation of planetary systems is the "Nebula Theory" whereby a cloud of interstellar gas accreted under it own gravitation, to form in stages, the protostar and the planets. While this is the leading theory about planet formation many questions (e.g the distribution of the angular momentum) are still under intense investigation. Many of the results related to the Nebula Theory were obtained through elaborate modeling and large scale numerical simulations. These involve, in general, thermodynamic considerations, magnetohydrodynamics modeling and turbulence [16]. In this paper however, we present an idealized steady state hydrodynamic analytic model which can capture the formation of ring structure in a self gravitating disk of stratified gas. To this end we show that under a wide range of this model parameters, matter density within the nebular cloud exhibits oscillations whose peaks are separated by an almost "empty space" viz.a ring structure as hypothesized by Laplace.

http://dx.doi.org/10.1016/j.apm.2015.04.034
0307-904X/

The basic assumptions of this model are that the interstellar cloud can be treated as a self gravitating, incompressible and stratified (viz. non-constant density) gas in which the particle velocities $|\mathbf{u}|$ are non-relativistic (i.e $|\mathbf{u}| \ll \bar{c}$ where $\bar{c}$ is the velocity of sound). It might be argued that the assumption of incompressibility and continuous mass distribution that we use in this model are not in line with the current scenarios for the evolution of interstellar gas cloud. Therefore the model presented here should viewed as an **idealization**. In particular the assumption of incompressibility can be justified (partially) only if the rate of change in the density is "negligible".

The plan of the paper is as follows: In Section 2 we present the model equations. In Section 3 we discuss the physical meaning of the two parameter functions that appear in this model. In Section 4 we present various radial solutions of the model equations (i.e. models where the material density is a function of the radial distance from the protostar). Both analytical and numerical results of this model are presented in this section. In Section 5 we discuss non-radial density distributions and end up in Section 6 with a summary and conclusions.

2. The model

The basic equations that govern the steady state of non-relativistic self gravitating, incompressible, inviscid and stratified two dimensional gas (in the $x - y$ plane) are [10,17–19]

$$u_x + v_y = 0, \tag{2.1}$$

$$u\rho_x + v\rho_y = 0, \tag{2.2}$$

$$\rho(uu_x + vu_y) = -p_x - \rho\phi_x, \tag{2.3}$$

$$\rho(uv_x + vv_y) = -p_y - \rho\phi_y, \tag{2.4}$$

$$\nabla^2\phi = 4\pi G\rho, \tag{2.5}$$

where subscripts indicate differentiation with respect to the indicated variable, $\mathbf{u} = (u, v)$ is the gas velocity, ρ is its density, p is the pressure, ϕ is the gravitational potential and G is the gravitational constant.

In this set of Eq. (2.1) is the incompressibility equation, (2.2) represents mass conservation while (2.3) and (2.4) are the momentum equations. Finally (2.5) is the equation for the gravitational potential.

We can nondimensionalize these equations by introducing the following scalings

$$x = L\tilde{x}, \quad y = L\tilde{y}, \quad u = U_0\tilde{u}, \quad v = U_0\tilde{v}, \quad \rho = \rho_0\tilde{\rho}, \quad p = \rho_0 U_0^2\tilde{p}, \quad \phi = U_0^2\tilde{\phi}, \quad \widetilde{G} = \frac{G\rho_0 L^2}{U_0^2}. \tag{2.6}$$

where L, U_0, ρ_0 are some characteristic length,velocity and mass density respectively that characterize the stellar gas at hand. Substituting these scalings in Eqs. (2.1)–(2.5) and dropping the tildes these equations remain unchanged (but the quantities that appear in these equations become non-dimensional)

In view of Eq. (2.1) we can introduce a stream function ψ so that

$$u = \psi_y, \quad v = -\psi_x. \tag{2.7}$$

Using this stream function we can rewrite Eq. (2.2) as [20,18]

$$J\{\rho, \psi\} = 0, \tag{2.8}$$

where for any two (smooth) functions f, g

$$J\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}. \tag{2.9}$$

Eq. (2.8) implies that the functions ρ, ψ are dependent on each other and we can express each of them in terms of the other. Thus we can write ψ as $\psi(\rho)$ or ρ as $\rho(\psi)$.

Using these facts in the momentum Eqs. (2.3) and (2.4) we obtain after some (intricate) algebra [10,21,19] that Eqs. (2.1)–(2.5) can be reduced to a system of two equations

$$h(\rho)^{1/2}\nabla \cdot (h(\rho)^{1/2}\nabla\rho) + \phi = S(\rho) \tag{2.10}$$

and Eq. (2.5). We observe the (2.10) contains two "parameter functions" $h(\rho)$ and $S(\rho)$ whose determination and physical meaning will be discussed in the next section.

A further simplification of Eqs. (2.5) and (2.10) is possible if we assume that $\rho = \rho(r)$ and $\phi = \phi(r)$ (where $r^2 = x^2 + y^2$). In this case these equations reduce to following coupled system of ordinary differential equations:

$$\rho'' = -\frac{\rho'}{r} - \frac{1}{h}\frac{dh}{d\rho}(\rho')^2 + \frac{1}{h^2}[S(\rho) - \phi], \tag{2.11}$$