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Numerical method for solving arbitrary linear differential equations using a set of orthogonal basis functions and operational matrix

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ABSTRACT

This article presents a numerical method for solving ordinary linear differential equations of arbitrary order and coefficients. For this purpose, block-pulse functions (BPFs) as a set of piecewise constant orthogonal functions are used. By the BPFs vector forms and operational matrix of integration, solving the differential equation is reduced to solve a linear system of algebraic equations. Some problems are numerically solved by the proposed method to illustrate its generality and computational efficiency for solving an arbitrary linear differential equation. For further evaluation, mean-absolute errors and running times associated with the method are given to show that the method is convergent and runs in a reasonable time.

1. Introduction

In recent decades, BPFs have been finding an important role in numerical analysis. Especially, valuable efforts have been spent, by researchers, on introducing novel ideas for numerical solution of various functional equations by using properties of these functions. A numerical direct method is presented in [1] for solution of some electromagnetic scattering problems using BPFs. Ref. [2] presents an expansion-iterative technique based on these functions for solving Fredholm integral equations of the first kind arising in one- and two-dimensional electromagnetic scattering. A similar concept is used in [3] to establish a numerical expansion-iterative method for solving Volterra integral equation of the first kind. BPFs are respectively used in [4,5] for implementation of collocation method for analysis of some perfectly conducting and resistive scatterers. Ref. [6] presents a direct method for solution of Volterra integral equation of the first kind based on BPFs and their operational matrix of integration. Operational matrices of BPFs are also used in [7] to solve Volterra integral and integro-differential equations of convolution type by finding numerical inversion of Laplace transform. Further information as to BPFs may be found in [8–13].

A great deal of interest has been focused on the solution of ordinary linear differential equations. A modification of Adomian decomposition method has been introduced in [14] for solving second-order ordinary differential equations. Also, [15] uses homotopy perturbation method to solve specific second-order ordinary differential equations. A numerical method for solving some differential equations is presented in [16] in which the operational matrix of integration of Chebyshev wavelets basis and its product operational matrix are used. Ref. [17] describes a method for numerical solution

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of linear integral equations based on Lagrange interpolation and then it is extended to be applied in solving linear integro-differential and differential equations. A method is presented in [18] for obtaining the particular solution of ordinary differential equations with constant coefficients and an explicit formula of the particular solution is derived from the use of an upper triangular Toeplitz matrix. Ref. [19] proposes a numerical scheme to solve the delay differential equations of pantograph type. The method consists of expanding the required approximate solution as the elements of the shifted Chebyshev polynomials.

Some of the above mentioned methods show a reasonable computational efficiency, however all of them have major limitations. For instance, they have been proposed for the solution of linear differential equations of specific *orders* or specific *coefficients*. Also, some of the methods have essentially been formulated for special *types* of differential equations. It is the main aim of this article to present a method that has the potential to be free of such limitations, while having a reasonable computational efficiency.

This article proposes a numerical method for solving linear ordinary differential equations of arbitrary order and coefficients. For this purpose, BPFs as a set of piecewise constant orthogonal basis functions are used to approximate the solution, its derivatives, and the equation coefficients. By using the BPFs operational matrix of integration, the BPFs coefficients vector of the solution and that of its various derivatives are expressed in terms of the BPFs coefficients vector of the highest order derivative and the initial conditions vectors, that results in a linear system of algebraic equations. Solving this system gives the BPFs coefficients vector of the highest order derivative and, accordingly, an approximate solution for the differential equation is obtained. The main advantages of the proposed method are as follows:

- The formulation of the method is quite general, without limitation or restriction. Therefore, it can be used for numerically solving an extensive variety of linear ordinary differential equations of arbitrary order and coefficients.
- The accuracy of the method is satisfactory.
- The running times of its algorithm, even for high degrees of approximation, are in a reasonable range.
- The method is convergent. Its error decreases as the grid-spacing is reduced. Thus, one can increase the degree of approximation until the results settle down to a desired accuracy.
- The algorithm is simple and clear to use and can be implemented easily.

The organization of this article is as follows. A review on BPFs and their properties is provided in Section 2. The representation error in approximation by BPFs is surveyed in Section 3 to get an error bound and evaluate the order of convergence. In Section 4, some other types of piecewise constant orthogonal functions including Rademacher, Walsh, and Haar functions are studied in conjunction with BPFs. Section 5 presents the numerical method for solving arbitrary linear ordinary differential equations by using the BPFs vector forms and operational matrix of integration. Some test problems are numerically solved in Section 6 by the proposed method and the related numerical results are given. There will be extensive varieties of orders, coefficients, types, and solutions associated with the test problems to illustrate the generality and computational efficiency of the proposed method. For further evaluation of the method, we present in Section 7 the running times of its algorithm for solving the presented test problems and also the mean-absolute errors associated with the method. Finally, conclusions will be given in Section 8.

2. Block-pulse functions and their vector forms

2.1. Definition

An m-set of BPFs is defined over a real interval $[0, H)$ as [1–3,6]

$$\varphi_i(t) = \begin{cases} 1, & \frac{iH}{m} \leqslant t < \frac{(i+1)H}{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $i = 0, 1, \ldots, m-1$, with a positive integer value for m. Also, consider $h = H/m$, and φ_i is the ith BPF. Here, we assume that $H = 1$, so BPFs are defined over $[0, 1)$, and $h = 1/m$. A set of BPFs over $[0, 1)$ for $m = 4$ is shown in Fig. 1.

There are some properties for BPFs, the most important properties are disjointness, orthogonality, and completeness.

Let us consider the first m terms of BPFs and write them concisely as an m-vector

$$\Phi(t) = [\,\varphi_0(t) \quad \varphi_1(t) \quad \ldots \quad \varphi_{m-1}(t)\,]^T, \quad t \in [0, 1), \tag{2}$$

where, superscript T indicates transposition. The above representation and disjointness property follows

$$\Phi(t)\Phi^T(t)V = \tilde{V}\Phi(t), \tag{3}$$

where V is an m-vector and $\tilde{V} = \mathrm{diag}(V)$. Moreover, it can be clearly concluded that for any $m \times m$ matrix B

$$\Phi^T(t)B\Phi(t) = \hat{B}^T\Phi(t), \tag{4}$$

where $\hat{B}$ is an m-vector with elements equal to the diagonal entries of matrix B. Also,