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The quasi-reversibility regularization method for identifying the unknown source for time fractional diffusion equation ☆

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ABSTRACT

The inverse problem of determining the unknown source for a fractional diffusion equation is studied. This problem is ill-posed in the sense of Hadamard, i.e., small changes in the measured data can blow up the solution. The quasi-reversibility regularization method is applied to solve it. Convergence estimates are presented under an *a priori* parameter choice rule and an *a posteriori* parameter choice rule, respectively. Numerical examples are given to show that the regularization method is effective and stable.

1. Introduction

The time-fractional diffusion equation is usually used to describe the anomalous diffusion phenomena which show many different aspects from normal ones. By an argument similar to the derivation of the classical diffusion equation from Brownian motion, one can derive a fractional diffusion equation from continuous-time random walk. The parameter determination in a diffusion equation from the over specified data plays an important role in applied mathematics, physics and engineering. An inverse problem of finding an unknown time-dependent coefficient in a parabolic partial differential equation has been considered in [1–6]. Inverse source problems also arise in many branches of science and engineering, e.g. heat conduction, crack identification, electromagnetic theory, geophysical prospecting and pollutant detection. For the heat source identification, there have been a large number of research results for different forms of heat source [7–18]. To the best of authors knowledge, there were few papers for identifying an unknown source for a fractional diffusion equation by regularization method. In [19], using the analytic continuation and Laplace transform, the authors proved the uniqueness of the identification of the unknown source dependent only on spatial variable for the fractional diffusion equation in a bound domain. In [20], using the coupled method, the authors identified the unknown source for the spatial fractional diffusion equations. In [21], the authors identified the unknown source for the time fractional diffusion equation using the mollification method.

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In this paper, we consider an inverse source problem for the time-fractional diffusion equation as follows:

$$\begin{cases} \partial_{0^+}^{\alpha} u(x,t) - u_{xx}(x,t) = f(t), & x > 0,\ t > 0,\ 0 < \alpha < 1, \\ u(x,0) = 0, & x \geqslant 0, \\ u(0,t) = 0, & t \geqslant 0, \\ u(x,t)|_{x\to\infty} bounded, & t \geqslant 0, \\ u(1,t) = g(t), & t \geqslant 0, \end{cases} \tag{1.1}$$

where the time-fractional derivative $\partial_{0^+}^{\alpha} u(x,t)$ is the Caputo fractional derivative of order α $(0 < \alpha \leqslant 1)$ defined by ([22])

$$\partial_{0^+}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \quad 0 < \alpha < 1, \tag{1.2}$$

$$\partial_{0^+}^{\alpha} u(x,t) = \frac{\partial u(x,t)}{\partial t}, \quad \alpha = 1, \tag{1.3}$$

If the function $f(t)$ is known, we can work out $u(x,t)$ by the initial boundary value. However, the problem here is that the source function $f(t)$ is unknown, which needs to be decided by some additional data. The additional data used in this study is the observation at a final moment $x = 1$. The data $g(t)$ is based on (physical) observation, and we assume the measured data $g_{\delta}(t) \in L^2(\mathbb{R})$ satisfies

$$\|g - g_{\delta}\| \leqslant \delta. \tag{1.4}$$

We define all functions to be zero for $t < 0$ in order to analyze the problem (1.1) in $L^2(\mathbb{R})$. The Fourier transform of function $f(t)$ is defined as follows:

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi t} f(t) dt. \tag{1.5}$$

The application of the Fourier transform technique to problem (1.1) with respect to the variable t yields the following problem in frequency space:

$$\begin{cases} (i\xi)^{\alpha} \hat{u}(x,\xi) - \hat{u}_{xx}(x,\xi) = \hat{f}(\xi), & x > 0,\ \xi \in \mathbb{R}, \\ \hat{u}(0,\xi) = 0, & \xi \in \mathbb{R}, \\ \hat{u}(x,\xi)|_{x\to\infty} bounded, & \xi \in \mathbb{R}, \\ \hat{u}(1,\xi) = \hat{g}(\xi), & \xi \in \mathbb{R}. \end{cases} \tag{1.6}$$

We can easily get the solution of problem (1.6)

$$\hat{f}(\xi) = \frac{(i\xi)^{\alpha}}{1 - e^{-(i\xi)^{\frac{\alpha}{2}}}} \hat{g}(\xi). \tag{1.7}$$

So

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} \frac{(i\xi)^{\alpha}}{1 - e^{-(i\xi)^{\frac{\alpha}{2}}}} \hat{g}(\xi) d\xi, \tag{1.8}$$

where

$$(i\xi)^{\alpha} = \begin{cases} |\xi|^{\alpha} \left(\cos\left(\frac{\alpha\pi}{2}\right) + i \sin\left(\frac{\alpha\pi}{2}\right)\right), & \xi \geqslant 0, \\ |\xi|^{\alpha} \left(\cos\left(\frac{\alpha\pi}{2}\right) - i \sin\left(\frac{\alpha\pi}{2}\right)\right), & \xi < 0, \end{cases} \tag{1.9}$$

$$(i\xi)^{\frac{\alpha}{2}} = \begin{cases} |\xi|^{\frac{\alpha}{2}} \left(\cos\left(\frac{\alpha\pi}{4}\right) + i \sin\left(\frac{\alpha\pi}{4}\right)\right), & \xi \geqslant 0, \\ |\xi|^{\frac{\alpha}{2}} \left(\cos\left(\frac{\alpha\pi}{4}\right) - i \sin\left(\frac{\alpha\pi}{4}\right)\right), & \xi < 0. \end{cases} \tag{1.10}$$

From the right side of (1.7) or (1.8), we can see that

$$\left| \frac{(i\xi)^{\alpha}}{1 - e^{-(i\xi)^{\frac{\alpha}{2}}}} \right| = \frac{|\xi|^{\alpha}}{\sqrt{1 - 2e^{-|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\pi}{4}\alpha\right)} \cos\left(|\xi|^{\frac{\alpha}{2}} \sin\left(\frac{\pi}{4}\alpha\right)\right) + e^{-2|\xi|^{\frac{\alpha}{2}} \cos\left(\frac{\pi}{4}\alpha\right)}}} \to \infty, \quad |\xi| \to \infty. \tag{1.11}$$

Since our solution $f(x)$ is assumed to be in $L^2(\mathbb{R})$, the exact data function $\hat{g}(\xi)$ must decay fast. But the measured data function $g_{\delta}(t)$ which only belongs to $L^2(\mathbb{R})$, does not posses such a decay property in general. Thus if we try to obtain the source $f(t)$, high frequency components in the error are magnified and can destroy the solution. It is impossible to solve the problem (1.1) by using classical methods. In the following section, we will use the quasi-reversibility regularization method to deal