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Modeling renewal processes in fuzzy decision system

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Under expected value of fuzzy variable and continuous Archimedean triangular norms, this paper discusses a renewal process and a renewal reward process for T-independent *L-R* fuzzy variables in fuzzy decision systems. First, a renewal process with T-independent *L-R* fuzzy interarrival times is discussed, some limit theorems on renewal variable, average renewal time, and long-term renewal rate in (fuzzy) measure are obtained, and a fuzzy elementary renewal theorem is proved for the limit of the long-term expected renewal rate. Second, a renewal reward process with T-independent *L-R* fuzzy interarrival times and rewards is discussed, a limit theorem on reward rate in (fuzzy) measure is derived, and a fuzzy renewal reward theorem is proved for the limit value of expected reward rate. Finally, the comparison with stochastic counterparts shows an interesting and reasonable homology in convergence mode and limit value between the results obtained in fuzzy renewal processes and the corresponding results in stochastic renewal processes, though they build on two essentially different mathematical cornerstones, possibility theory and probability theory, respectively.

1. Introduction

Stochastic renewal theory has been well developed based on probability theory. For a stochastic renewal process, the interarrival times and rewards are assumed to be independent and identically distributed random variables. Under this assumption, elementary renewal theorem, key renewal theorem, delayed renewal theorem, and renewal reward theorem as key results in the renewal theory have been developed in the past decades (see [1,2]), and some recent applications can be found in [3–6].

As for renewal processes in fuzzy decision systems, information on the lifetime and cost variables of a system are usually fuzzily or ambiguously imprecise which is intrinsically different from the probabilistic variability (see [7–12]). Therefore, it is more reasonable to utilize possibility theory, which serves as the counterpart of probability theory to deal with fuzziness, to study such renewal processes. The theory of possibility was introduced by Zadeh [12] to study on the behaviors of fuzzy events, and had been further developed by a number of researchers such as [8,11,13,14] from the perspective of fuzzy or nonadditive measures. As a core concept in possibility theory, fuzzy variable is generally defined as a mapping from the possibility space to the set of real numbers (see [10,14]), and in order to compute the numerical characteristics of a fuzzy variable, an expected value of fuzzy variable was defined via Choquet integral (see [15]). The independence of fuzzy variables is another pivotal issue in the possibility theory. The classic independence of fuzzy variables is defined via minimum triangular

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norm (t-norm), namely, the min-independence (see [8]). Extending the *min* operator to a general t-norm $\top$ forms a generalized independence, namely, T-independence (see [13]). That is, different t-norms induce different T-independence operations. An increasing number of studies have been done in the domain of T-independent (also called "T-related" in the literature) fuzzy variables or fuzzy numbers (see [13,16–20]), for it provides a wider range of options for arithmetic operator selection towards a variety of applications, e.g., image processing [21,22], fuzzy logic [23], and measurement theory [24,25].

Fuzzy renewal processes in the context of possibility theory have not been studied until a decade ago, and only a few results have been obtained through different independence assumptions and problem settings. Among them, assuming the interarrival times and rewards are continuous and identically independent fuzzy variables, a fuzzy renewal process was discussed in [26], the authors proved a fuzzy elementary renewal theorem and a fuzzy renewal reward theorem by using the expected value of fuzzy variable. Going deeper with the results in [26], a fuzzy alternating renewal process was investigated in [27], and some properties were obtained on the asymptotic on-time and off-time behaviors per unit time. For both studies in [27,26], a key assumption is the min-independence of the fuzzy interarrival time sequences and reward sequences. On the other hand, under the assumption of continuous Archimedean t-norm based independence (minimum t-norm does not belong to Archimedean t-norms), a fuzzy renewal process was re-considered by Hong [28] in which the interarrival times and rewards are modeled as T-related *L-R* fuzzy variables, and obtained some initial results on long-term renewal and reward rates under necessity measure (the dual set function of possibility measure, see [8]), but no results have been achieved under expectation or any mean value operators.

Recall that a key role the stochastic elementary renewal theorem and renewal reward theorem play in stochastic renewal process (or even in the stochastic decision systems) is that they show the long-term behaviors of renewal and reward rates in the sense of expectation, which is crucial and far-reaching to the process-based decision systems. Unfortunately, in the fuzzy renewal processes with T-independent variables, the corresponding fuzzy renewal theorems for the long-term expected renewal rate and the long-term expected reward rate have not been well developed yet. To this end, our study discusses the fuzzy renewal processes with T-independent interarrival times and rewards in a comprehensive and consistent manner by investigating the limit behaviors of renewal and reward rates under both fuzzy measures (possibility and credibility) and expected values.

The rest of the paper is outlined as follows. Section 2 introduces some basic concepts and results of fuzzy variables, T-independence of fuzzy variables and convex hull of a real function. In Section 3, we discuss a fuzzy renewal process by assuming the interarrival times to be T-independent *L-R* fuzzy variables. Furthermore, a fuzzy renewal reward process with T-independent *L-R* fuzzy rewards is discussed in Section 4. In Section 5, a detailed comparison is presented in both theory and applications between the developed fuzzy renewal models and the classic stochastic renewal models. Finally, a conclusion is given in Section 6.

2. Preliminaries

2.1. Fuzzy variable

Given a universe Γ, let Pos be a set function defined on the power set $\mathcal{P}(\Gamma)$ of Γ. The set function Pos is said to be a possibility measure if it satisfies the following conditions

(Pos1) $\mathrm{Pos}(\emptyset) = 0$, and $\mathrm{Pos}(\Gamma) = 1$;

(Pos2) $\mathrm{Pos}\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I}\mathrm{Pos}(A_i)$ for any subclass $\{A_i, i \in I\}$ of $\mathcal{P}(\Gamma)$, where I is an arbitrary index set.

Based on possibility measure, a self-dual set function Cr, called *credibility measure*, can be formally defined as follows:

Definition 1 [15]. Let Pos be a possibility measure. The credibility measure is defined as

$$\mathrm{Cr}(A) = \frac{1}{2}\left(1 + \mathrm{Pos}(A) - \mathrm{Pos}(A^c)\right), \quad A \in \mathcal{P}(\Gamma), \tag{1}$$

where A^c is the complement of A.

The triplet $(\Gamma, \mathcal{P}(\Gamma), \mathrm{Cr})$ is called a *credibility space* [10]. A credibility measure has the following properties:

(1) $\mathrm{Cr}(\emptyset) = 0$, and $\mathrm{Cr}(\Gamma) = 1$.
(2) Monotonicity: $\mathrm{Cr}(A) \leqslant \mathrm{Cr}(B)$ for any $A, B \in \mathcal{P}(\Gamma)$ with $A \subset B$.
(3) Self-duality: $\mathrm{Cr}(A) + \mathrm{Cr}(A^c) = 1$ for any $A \in \mathcal{P}(\Gamma)$.
(4) Subadditivity: $\mathrm{Cr}(A \cup B) \leqslant \mathrm{Cr}(A) + \mathrm{Cr}(B)$ for any $A, B \in \mathcal{P}(\Gamma)$.