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# Sensitivity analysis for some inverse problems in linear elasticity via minimax differentiability

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ABSTRACT

In this article, we consider the inverse problem of recovering a piecewise constant Lamé parameters by a single boundary measurement. We also consider the geometric inverse problem of locating the interface where the jump of the parameters occurs. These problems turn out to an optimization problems by making use of the Kohn–Vogelius cost function. We rewrite the functional in a min–sup form and we use the differentiability of the min–sup combined with the function space parametrization and the function space embedding to get the optimality condition. These techniques allow us to avoid the differentiability of the states variables with respect to the shape or the Lamé parameters. We apply an iterative algorithm and we give some numerical results.



## 1. Introduction

This paper is devoted to the mathematical analysis of some inverse problems in linear elasticity, namely the identification of Lamé parameters and the interfaces where the jump of the parameters occurs, from boundary measurement. This type of inverse problems arises in practical situations like the design, the control of optimal industrial structures. In the case of medical imaging, inverse parameter estimation can potentially be used to determine properties and location of different tissue types while using minimally invasive technologies instead of dissecting the patient. Classifying the elastic properties of tissue and locating abnormalities can help to identify cancerous growth.

From the theoretical point of view, the inverse problem of Lamé parameters has been studied by several authors. In the two dimensional case and when the parameters are $C^\infty$ functions, Akamatsu, Nakamura and Steinberg [1], proved that for the case of full Cauchy data, one can recover the Taylor series of Lamé parameters on the boundary. This result was extended into higher dimensions [2]. For the case of full Cauchy data, Nakamura and Uhlmann [2] established that in the two dimensions, the Lamé coefficients are uniquely determined, assuming that they are sufficiently close to a pair of positive constants. Recently Imanuvilov and Yamamoto in [3] proved for the two dimensional case that the Lamé coefficient $\lambda$ can be recovered from partial Cauchy data if the coefficient $\mu$ is some positive constant. For the three dimensional case Nakamura and Uhlmann in [4,5] and Eskin and Ralston [6] proved uniqueness for both Lamé coefficients when $\mu$ is assumed to be close to a positive constant. The proofs in the above papers rely on construction of complex geometric optics solutions. The question of stability was addressed by Isakov, Wang and Yamamoto [7], they proved a Hölder and Lipschitz stability estimates of

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determining all coefficients of a dynamical Lamé system with residual stress, including the density Lamé parameters, and the residual stress, by three pairs of observations from the whole boundary or from a part of it. The geometric inverse problem was addressed by several authors, for more details the reader is referred to [8–14]. Nevertheless it is still a challenging problem both for the mathematical and numerical aspects.

In the absence of general analytic formulae, the inverse problems are usually solved by minimizing an objective function that measures the mismatch between the model predictions and the measurements. A central element in the minimization procedure is the calculation of the gradient of the objective function with respect to the variations in the shape of the interfaces or the parameters, which is a basic tool to obtain necessary conditions and to provide us with gradient information required by the gradient type optimization methods.

For the study of the optimization problems, the traditional method involves the computation of the state derivative with respect to the domain or the parameters. Usually such a derivative requires stronger regularity assumptions which are not satisfied in the elasticity problem with discontinuous Lamé coefficients. However, the state differentiability is not necessary in many cases, and even when the state variable is not differentiable. It is well established now that the sensitivity analysis (see [15–23]) remains a powerful tool to solve inverse problems. Therefore, avoiding the differentiability of the state is of a great importance in such situations and particularly for the problem under consideration.

In this work, we propose a Kohn–Vogelius type cost function. We express each of the optimization problems as a min max of suitable lagrangian functional. The characterization of the change in geometric domain is obtained by the velocity method [15,16]. Finally we use the theorem on the differentiability of a saddle point (i.e., a minimax) of such lagrangian functional with respect to a parameter which provides very powerful tools to obtain the gradient of the cost function by function space parametrization or function space embedding [16] without using the derivative of the state. The geometrical inverse problem presented in this paper extends the result presented in [24] to locate the jump of the conductivity for the Laplace equation.

The rest of the paper is restructured as follows. After introducing the formulation of the problem in Section 2. We perform the saddle point formulation of the problems (2.2) and (2.3) and the Lagrangian associated with the cost functional (2.4) in Section 3. Section 4 is devoted to the differentiability of the cost function with respect to Lamé parameters. In Section 5, we use the function space parametrization and the function space embedding to compute the shape derivative of the functional $J$. Section 6 is devoted to the characterization of the shape gradient. In Section 7 we present a gradient type algorithm to solve numerically the inverse problem related to the identification of an inclusion in a particular case and we use the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm to find numerically the Lamé coefficients. We end in the last section by a short conclusion.

## 2. Problem Formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$, where $\Gamma_N \cap \Gamma_D = \emptyset$. Suppose that the elastic medium $\Omega$ contains a single inclusion $\omega$ which is also a bounded Lipschitz domain. We suppose that $\mathrm{dist}(\partial\Omega, \partial\omega) > \delta$ for some positive constant $\delta$. Let the constants $(\lambda_e, \mu_e)$ denote the background Lamé coefficients, that are the elastic parameters in the absence of any inclusion. Suppose that $\omega$ has the pair of Lamé parameters $(\lambda_s, \mu_s)$ which is different from those of the background elastic body. It is always assumed that

$$\mu_e > 0, \quad \lambda_e + \mu_e > 0, \quad \mu_s > 0 \text{ and } \lambda_s + \mu_s > 0,$$

$$(\lambda_e - \lambda_s)(\mu_e - \mu_s) \geqslant 0, \quad \left((\lambda_e - \lambda_s)^2 + (\mu_e - \mu_s)^2\right) \neq 0.$$

We consider the inverse problems of recovering the interface $\partial\omega$ and the Lamé parameters $(\lambda_e, \mu_e, \lambda_s, \mu_s)$ from boundary measurement. Namely, given the deformation $u(x) = f$ on $\Gamma_N, u(x) = 0$ on $\Gamma_D$ (Dirichlet data) and the forces in the direction of the normal $\sigma(u)\nu = g$ on $\Gamma_N$ (Neumann data), then the inverse problems consists of locating the interface $\partial\omega$ and the Lamé parameters from the knowledge of the pair $(f, g)$.

For a given current density $g \in (L^2(\Gamma_N))^2$, the deformation $u$ satisfies the following problem:

$$\begin{cases} \mathrm{div}\sigma(u) = 0 & \text{in } \Omega, \\ \sigma(u)\nu = g & \text{on } \Gamma_N, \\ u = 0 & \text{on } \Gamma_D, \end{cases} \tag{2.1}$$

The stress tensor $\sigma(u)$ is related by the linearized strain tensor $\varepsilon(u)$ via the Hooke's law:

$$\sigma(u) = C : \varepsilon(u) = \sum_{i,j,k,l=1}^{2} C_{ijkl} \frac{\partial u_k}{\partial x_l}, \quad \varepsilon(u) = \frac{1}{2}(Du + (Du)^*).$$

The elasticity tensor is given by

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad i,j,k,l = 1,2,$$