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Analytical treatment of Volterra integro-differential equations of fractional order

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ABSTRACT

In this paper, Volterra integro-differential equations of fractional order is investigated by means of the variational iteration method. The fractional derivative is described in the Caputo sense. Moreover, stability and convergence of the proposed scheme are analyzed. Finally, some examples are presented to illustrate the theoretical results.

1. Introduction

In recent years, various kinds of analytical methods and numerical methods were used to solve fractional integro-differential equations. For instance we can mention the following papers. Rawashdeh [1] applied collocation method to study the integro-differential equations of fractional order, authors of [2] applied the Adomian decomposition method (ADM) to approximate solutions for fourth-order integro-differential equations of fractional order, Lepik [3] applied the Haar wavelet method to solve the fractional integral equations, authors of [4] applied fractional differential transform method to approximate solutions for integro-differential equations of fractional order.

In the present paper, we apply variational iteration method (VIM) [5–11] to solve Volterra integro-differential equations. This method is now widely used by many researchers to study linear and nonlinear problems. This method is employed in [12] to solve the Klein–Gordon partial differential equations. Authors of [13] applied the variational iteration method to solve the Lane–Emden differential equation. For more applications of the method the interested reader is referred to [14–17].

In this study, we consider Volterra integro-differential equations of fractional order of the form

$$\begin{cases} D^{\alpha}\xi(x) - \lambda\int_0^x k(x,t)\xi(t)dt = g(x), \\ \xi^{(i)}(0) = c_i, \quad i = 0,1,2,\ldots,n-1, \; n = [\alpha]+1. \end{cases}$$

where $g \in L^2([0,X])$, $k \in L^2([0,X]^2)$ are given functions, D^{α} is the fractional derivative of order α, and $\xi(x)$ is unknown function.

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The rest of the paper is organized as follows. In Section 2, we recall some basic definitions and properties of the fractional calculus theory. In Section 3, we construct an algorithm for solving Volterra integro-differential equations of fractional order by using the VIM. In Section 4, stability of the proposed approach is discussed. In Section 5, the convergence conditions of our proposed scheme are formulated and proved. In Section 6, some illustrative examples are given. Some concluding remarks are given in Section 7.

2. Preliminaries and notations

This section deals with some preliminaries and notations regarding fractional calculus. For more details see [18–32].

Definition 1. A real function $\xi(t),\ t > 0$, is said to be in the space $C_\alpha, \alpha \in \Re$, if there exists a real number $p\ (> \alpha)$, such that $\xi(t) = t^p \xi_1(t)$, where $\xi_1(t) \in C[0,\infty)$, and it is said to be in the space $C_\alpha^m, m \in \mathcal{N} \bigcup \{0\}$, if and only if $\xi^{(m)}(t) \in C_\alpha$.

Definition 2. The Mittag–Leffler function $E_\alpha(z)$ with $\alpha > 0$ is an extension of the exponential function which defined by the following series representation, valid in the whole complex plane

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad z \in \mathbb{C}. \tag{1}$$

Definition 3. The (left sided) Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $\xi(t) \in C_\alpha,\ \alpha \geqslant -1$, is defined as

$$I_{0_+}^\alpha \xi(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\xi(\tau)}{(t-\tau)^{1-\alpha}} d\tau, & \alpha > 0,\ t > 0, \\ \xi(t), \end{cases} \tag{2}$$

$$I_{0_+}^\alpha \xi(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\xi(x,s)}{(t-s)^{1-\alpha}} ds, \quad \alpha > 0,\ t > 0, \tag{3}$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 4. The (left sided) Riemann–Liouville fractional derivative of $\xi(t),\ \xi(t) \in C_{-1}^m,\ m \in \mathcal{N} \bigcup \{0\}$, of order α is defined as

$$D_{0_+}^\alpha \xi(t) = \frac{d^m}{dt^m} I_{0_+}^{m-\alpha} \xi(t), \quad m-1 < \alpha \leqslant m,\ m \in \mathcal{N}. \tag{4}$$

Definition 5. The Caputo fractional derivative of $\xi(t),\ \xi(t) \in C_{-1}^m,\ m \in \mathcal{N} \bigcup \{0\}$, is defined as

$$^cD_{0_+}^\alpha \xi(t) = \begin{cases} \left[I_{0_+}^{m-\alpha} \xi^{(m)}(t) \right], & m-1 < \alpha < m,\ m \in \mathcal{N}, \\ \frac{d^m}{dt^m} \xi(t), & \alpha = m, \end{cases} \tag{5}$$

$$^cD_{0_+}^\alpha \xi(x,t) = I_{0_+}^{m-\alpha} \frac{\partial^m \xi(x,t)}{\partial t^m}, \quad m-1 < \alpha < m, \tag{6}$$

$$^cD_{0_+}^\alpha (D_{0_+}^m \xi(t)) = {}^cD_{0_+}^{\alpha+m} \xi(t), \quad m = 0, 1, \ldots, n-1 < \alpha < n. \tag{7}$$

Property. *Assume that the continues function $\xi(t)$, has a fractional derivative of order α, then we have*

$$^cD_{0_+}^\alpha (I_{0_+}^\beta \xi(t)) = \begin{cases} I_{0_+}^{\beta-\alpha} \xi(t), & \alpha < \beta, \\ \xi(t), & \alpha = \beta, \\ D_{0_+}^{-\beta+\alpha} \xi(t), & \alpha > \beta, \end{cases} \tag{8}$$

$$I_{0_+}^\alpha ({}^cD_{0_+}^\alpha \xi(t)) = \xi(t) - \sum_{k=0}^{m-1} \xi^{(k)}(0^+) \frac{t^k}{k!}, \quad m-1 < \alpha \leqslant m,\ m \in \mathcal{N}. \tag{9}$$