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Variational iteration method as a kernel constructive technique

Guo-Cheng Wu [a], Dumitru Baleanu [b,c,*], Zhen-Guo Deng [d]

[a] College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641112, PR China
[b] Department of Mathematics and Computer Sciences, Cankaya University, 06530 Balgat, Ankara, Turkey
[c] Institute of Space Sciences, Magurele-Bucharest, Romania
[d] School of Mathematics and Information Science, Guangxi University, Nanning 530004, PR China

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ABSTRACT

The variational iteration method newly plays a crucial role in establishing new integral equations. The Lagrange multipliers of the method serve kernel functions of the Volterra integral equations. A concept of an optimal integral equation is proposed. Then nonlinear examples are used to show the strategy's efficiency.

1. Introduction

The integral equation is an old topic which can date back to the 14th century. It arises frequently in many applied areas [1–4]. Up to now, integral equations become an important branch of modern mathematics. Various numerical methods have been developed to find the approximate solutions and several excellent monographs [5–9] are available.

When we solve a given nonlinear differential equation numerically, to establish equivalent integral equations is a crucial step and then numerical methods can be applied directly. However, a question may arise: how many equivalent integral equations can one differential equation have and whether there is any method searching all possible equivalent integral equations?

In this study, the variational iteration method (VIM) [10,11] is adopted to construct iteration schemes. Together with the convergence conditions, various integral equations are obtained and the optimal one then can be chosen for numerical and analytical methods. Several nonlinear examples are used to illustrate this purpose.

2. Variational Iteration method and the equivalent integral equations

The Lagrange multiplier method has been extensively used in mathematics physics and general mechanics. He [10,11] initially introduced the idea of the Lagrange multiplier to construct the correction functional when solving differential equations and developed the VIM. The method has been applied to nonlinear equations [12–19] as well as the fractional calculus [10,20–22].

* Corresponding author at: Department of Mathematics and Computer Sciences, Cankaya University, 06530 Balgat, Ankara, Turkey.
E-mail addresses: wuguocheng@gmail.com (G.-C. Wu), dumitru@cankaya.edu.tr (D. Baleanu).

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Let's illustrate the basic idea of the method. Consider the following ordinary differential equation (ODE) with Cauchy conditions

$$\frac{d^m u}{dt^m} + R[u] + N[u(t)] = g(t), \quad u^{(k)}(0) = c_k, \quad k = 0, \ldots, \quad m-1, \quad m \in Z^+, \tag{1}$$

where the linear term $R[u] = a_1 \frac{d^{m-1}u}{dt^{m-1}} + \cdots + a_m u$, $a_1, \ldots, a_m$ are constants, N is a nonlinear operator and $g(t)$ is a given continuous function.

According to the VIM [10,11], construct the iteration formula as

$$u_{n+1} = u_n + \int_0^t \lambda(t,\tau)\left(\frac{d^m u_n}{d\tau^m} + R[\tilde{u}_n] + N[\tilde{u}_n] - g(\tau)\right)d\tau. \tag{2}$$

where $\tilde{u}_n$ is a restricted variation implying $\delta\tilde{u}_n = 0$. The weighted function $\lambda(t,\tau)$ is called the Lagrange multiplier and some techniques [18,20–23] have been suggested.

For Eq. (1), one of the Lagrange multipliers in [18] reads

$$\lambda(t,\tau) = \frac{(-1)^m(\tau - t)^{m-1}}{(m-1)!}, \tag{3}$$

from which Eq. (2) is rewritten as

$$\begin{cases} u_{n+1} = u_0 + \int_0^t \frac{(-1)^m(\tau-t)^{m-1}}{(m-1)!}(R[u_n(\tau)] + N[u_n(\tau)] - g(\tau))d\tau, \\ u_0 = u(0) + u'(0)t + \cdots + \frac{u^{(m-1)}(0)t^{m-1}}{(m-1)!}. \end{cases} \tag{4}$$

Note that u_n is the n -th order approximate solution which tends to the exact solution u of (1) for $n \to \infty$.

The proof of the convergences used the Banach's fixed point theorem has been discussed in [24]. Consequently, the classical integral equation can be established for Eq. (1)

$$\begin{cases} u = f(t) + \int_0^t \frac{(-1)^m(\tau-t)^{m-1}}{(m-1)!}(R[u(\tau)] + N[u(\tau)] - g(\tau))d\tau, \\ f(t) = u_0 = \sum\limits_{k=0}^{m-1} c_k \frac{t^k}{k!}. \end{cases} \tag{5}$$

The method above indeed provides an efficient tool to construct an equivalent integral equation. On the other hand, since there are various choices of the Lagrange multipliers, new integral equations can be derived.

3. Integral equations for nonlinear differential equations

Example 3.1. Consider the nonlinear equation of second order,

$$\frac{d^2u}{dt^2} + 2\frac{du}{dt} + u + u^2 = 0, \quad u(0) = 0, \quad u'(0) = 1. \tag{6}$$

According to the VIM, firstly, we can construct the iteration formulae as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_1(t,\tau)\left(\frac{d^2u_n}{d\tau^2} + 2\frac{d\tilde{u}_n}{d\tau} + \tilde{u}_n + \tilde{u}_n^2\right)d\tau, \tag{7}$$

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_2(t,\tau)\left(\frac{d^2u_n}{d\tau^2} + 2\frac{du_n}{d\tau} + \tilde{u}_n + \tilde{u}_n^2\right)d\tau, \tag{8}$$

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_3(t,\tau)\left(\frac{d^2u_n}{d\tau^2} + 2\frac{du_n}{d\tau} + u_n + \tilde{u}_n^2\right)d\tau, \tag{9}$$

and

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_4(t,\tau)\left(\frac{d^2u_n}{d\tau^2} + 2\frac{d\tilde{u}_n}{d\tau} + u_n + \tilde{u}_n^2\right)d\tau, \tag{10}$$

The different linear terms are restricted in (7)–(9), such as $\frac{d\tilde{u}_n}{d\tau}$ and $\tilde{u}_n$ in Eq. (7), $\tilde{u}_n$ in Eq. (8) and $\frac{d\tilde{u}_n}{d\tau}$ in Eq. (10). The differences generate different Lagrange multipliers, $\lambda_1(t,\tau)$,. . .,$\lambda_4(t,\tau)$.