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Jacobi spectral method for the second-kind Volterra integral equations with a weakly singular kernel

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ABSTRACT

The Jacobi spectral Galerkin method for Volterra integral equations of the second kind with a weakly singular kernel is proposed in this paper. We provide a rigorous error analysis for the proposed method, which indicates that the numerical errors (in the $L^2_{\omega_{\alpha,\beta}}$-norm and the L^∞-norm) will decay exponentially provided that the source function is sufficiently smooth. Numerical examples are given to illustrate theoretical results.

1. Introduction

As we know, VIEs are models of evolutionary problems with memory arising in diverse science and engineering applications, and in recent years, a large body of literatures has been devoted to numerical solutions of VIEs (see, e.g.,[1–8]). Among existing methods, numerical schemes based on Taylor's expansions or quadrature formulas have been frequently used. The block-by-block method, the multistep method, and the implicit Runge–Kutta method, which can be systematically designed, often provide accurate approximations.

Over the years, the spectral method has become increasingly popular and been widely used in spatial discretizations of PDEs owing to its high order of accuracy(cf [9–13]). Since the spectral method is fully capable of solving problems with history dependence, a global spectral method could be a good candidate for numerical VIEs. Some work has been done along this line, and we particularly point out that in the literature [8], T. Tang, X. Xu and J. Chen first proposed and analyzed a spectral collocation method for Volterra integral equation with a smooth kernel. Subsequently, Y. Chen and T. Tang developed the spectral collocation method for Volterra integral equation with weakly singular kernel in [1,2]. However, the spectral-collocation method is similar to the finite-difference approach. It makes use of values of interpolation points to present coefficients of expanded form of the numerical solution, and as a result its computing scheme is complex and the corresponding error analysis is more tedious. So to find a simple and efficient method is very meaningful.

In this paper, we propose a kind of novel algorithm for the Volterra integral equations of the second kind with a weakly singular kernel, which is called the Jacobi spectral method, and it differs from the spectral-collocation method and has several advantages. Firstly, Although both the spectral method and the spectral-collocation algorithm possess the spectral accuracy, in the spectral method, we put the approximation scheme under the general inner product type framework and take advantage of the property of orthogonal polynomials sufficiently, the results are that the computing schemes of the

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spectral method are more simple and the relevant convergence theories, as will be seen from Sections 2,4, are cleaner and more reasonable than those obtained in [1,2]. Secondly, compared with the finite-difference method,etc., the spectral method possesses high accuracy. Finally, the numerical solution is represented by function form, so it can simulate more entirely the global property of exact solution and provides more information above the structures of exact solution.

The paper is organized as follows. In Section 2, we introduce the Jacobi spectral Galerkin approaches for the Volterra integral equations. Some preliminaries and useful lemmas are provided in Section 3. In Section 4, the convergence analysis is given. We prove the error estimates in the $L^2_{\omega_{\alpha,\beta}}$-norm and L^∞-norm. The numerical experiments are carried out in Section 5,6, which will be used to verify the theoretical results obtained in Section 4. The final section contains conclusions and remarks.

2. Jacobi spectral Galerkin method

In this section, we formulate the Jacobi spectral schemes for the following problem

$$y(x) = \widetilde{f}(x) + \int_0^x (x-\tau)^{-\mu}\widetilde{K}(x,\tau)y(\tau)d\tau, \qquad 0<\mu<1, 0<x<T, \tag{2.1}$$

where the unknown function $y(x)$ is defined in $0 < x \leqslant T < \infty, \widetilde{f}(x)$ is a given source function and $\widetilde{K}(x,\tau)$ is a given kernel.

The numerical treatment of (2.1) is not simple, mainly due to the fact that the solutions of (2.1) usually have a weak singularity at $x=0$, even when the inhomogeneous term $\widetilde{f}(x)$ is regular.

It is known that the singular behavior of the exact solution makes the direct application of the spectral methods difficult. To overcome this difficulty, we first make the following linear transformation

$$\bar{u}(x) = x^{m+\mu-1}y(x)$$

The weakly singular problem (2.1) becomes:

$$\bar{u}(x) = \bar{f}(x) + x^{m+\mu-1}\int_0^x \tau^{1-m-\mu}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)\bar{u}(\tau)d\tau, \qquad 0<\mu<1, 0\leqslant x\leqslant T, \tag{2.2}$$

where

$$\bar{f}(x) = x^{m+\mu-1}\widetilde{f}(x)$$

and $\bar{u}(x)$ is m-th smooth solution of Eq. (2.2).

For the sake of applying the theory of orthogonal polynomials conveniently, by the linear transformation

$$x = \frac{T(1+t)}{2}, \quad \tau = \frac{T(1+s)}{2}$$

letting

$$u(t) = \bar{u}(\frac{T(1+t)}{2}), \quad g(t) = \bar{f}(\frac{T(1+t)}{2}), \quad \Lambda = [-1,1],$$

the weakly singular problem (2.2) can be rewritten as follows:

$$u(t) = g(t) + \int_{-1}^t (1+s)^{-\mu}(t-s)^{-\mu}K(t,s)u(s)ds, \quad t\in\Lambda, \tag{2.3}$$

where

$$K(t,s) = \left(\frac{T}{2}\right)^{1-\mu}(1+s)^{1-m}(1+t)^{m+\mu-1}\widetilde{K}\left(\frac{T(1+t)}{2},\frac{T(1+s)}{2}\right).$$

Next, we describe the Jacobi spectral method. For this purpose, Let $\omega_{\alpha,\beta} = (1-t)^\alpha(1+t)^\beta$ be a weight function in the usual sense, for $\alpha,\beta > -1$. $J_k^{\alpha,\beta}(t), k=0,1,\ldots$, denote the Jacobi polynomials. The set of Jacobi polynomials $\{J_k^{\alpha,\beta}\}_{k=0}^\infty$forms a complete $L^2_{\omega_{\alpha,\beta}}(-1,1)$-orthogonal system.

We first define a linear integral operator $\mathcal{M} : C(\Lambda)\longrightarrow C(\Lambda)$ by

$$(\mathcal{M}u)(t) = (t+1)^{\mu+m-1}\int_{-1}^t (s+1)^{1-m-\mu}(t-s)^{-\mu}\left(\frac{T}{2}\right)^{1-\mu}\widetilde{K}(t,s)u(s)ds = \int_{-1}^1 (1-\theta^2)^{-\mu}\left(\frac{t+1}{2}\right)^{1-2\mu}K(t,\theta)u(\theta)d\theta, \tag{2.4}$$

where $t\in(-1,1)$ and $s = s(t,\theta) = \frac{t-1}{2} + \frac{t+1}{2}\theta, \quad \theta\in[-1,1]$.Then, problem (2.3) reads: Find $u = u(t)$, such that

$$u(t) = g(t) + (\mathcal{M}u)(t) \tag{2.5}$$

and its weak form is to find $u\in L^2_{\omega_{\alpha,\beta}}(\Lambda)$, such that

$$(u,v)_{\omega_{\alpha,\beta}} = (g+\mathcal{M}u, v)_{\omega_{\alpha,\beta}}, \quad \forall v\in L^2(\Lambda),$$

where $(\cdot,\cdot)_{\omega_{\alpha,\beta}}$ denotes the usual inner product in the $L^2_{\omega_{\alpha,\beta}}$-space.