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Finite size Lyapunov exponent at a saddle point

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ABSTRACT

A simple stochastic system is considered modeling Lagrangian motion in a vicinity of a hyperbolic stationary point in two dimensions. We address the dependence of the Finite Size Lyapunov Exponent (FSLE) λ on the diffusivity D and the direction of the initial separation θ. It is shown that there is an insignificant difference between the curves $\lambda = \lambda(\theta)$ for pure dynamics ($D = 0$) and for infinitely large noise ($D = \infty$). For small D a well known boundary layer asymptotic is employed and compared with numerical results.

1. Introduction

In recent years FSLE has become a popular tool for investigating mixing in ocean and atmospheric flows, e.g. [1,2]. In theoretical works the focus was mostly on the scaling of $\lambda(\delta)$ as a function of the initial separation magnitude δ for flows close to isotropic, e.g. [3–6].

In the presented work we consider an extremely anisotropic flow and first suggest a closed solution for the case of infinitely large noise. It is found that the anisotropy in λ is almost not affected by an infinite diffusivity.

To analyze the case of small D we use the boundary layer approach, e.g. [7], as well as some heuristic arguments, and compare results with direct numerical computations of λ.

Next we proceed to exact formulations.

Let $\mathbf{X}(t, \mathbf{a})$ be a Lagrangian trajectory starting at $\mathbf{a}$ in a stochastic velocity field $\mathbf{u}(t, \mathbf{x})$

$$\dot{\mathbf{X}} = \mathbf{u}(t, \mathbf{X}), \quad \mathbf{X}(0) = \mathbf{a}, \tag{1}$$

and define the separation process

$$\mathbf{Z}(t, \mathbf{a}, \Delta\mathbf{a}) = \mathbf{X}(t, \mathbf{a} + \Delta\mathbf{a}) - \mathbf{X}(t, \mathbf{a}).$$

Let $\tau = \tau(\mathbf{a}, \Delta\mathbf{a}, \alpha)$ be the first moment when the separation hits $\alpha|\Delta\mathbf{a}|$ where $\alpha > 1$ is a prescribed threshold

$$\tau = \inf\{t > 0 : \quad |\mathbf{Z}(t)| \geqslant \alpha|\Delta\mathbf{a}|\}.$$

Following to [3,4] define the forward FSLE by

$$\lambda^+ = \frac{\ln \alpha}{\langle \tau \rangle},$$

where the angels mean the ensemble averaging, and in a similar way define the backward FSLE λ^- by replacing in (1) $\mathbf{u}(t, \mathbf{x})$ with $-\mathbf{u}(t, \mathbf{x})$. Finally, FSLE itself is defined by

$$\lambda = \lambda(\mathbf{a}, \Delta\mathbf{a}, \alpha) = \lambda^+ - \lambda^-.$$

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The problem of analytical studying λ for a general stochastic velocity field is extremely hard, in particular because the separation equation

$$\dot{\mathbf{Z}} = \mathbf{u}(t, \mathbf{X}(t, \mathbf{a} + \Delta\mathbf{a})) - \mathbf{u}(t, \mathbf{X}(t, \mathbf{a})),$$

is not closed. To advance we make a few assumptions under which it allows for a closure. First, assume

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{U}(\mathbf{x}) + \mathbf{u}'(t, \mathbf{x}), \tag{2}$$

where $\mathbf{U}(\mathbf{x}) = \mathbf{Gx}$ is a deterministic velocity field with constant matrix $\mathbf{G}$ (shear tensor) and $\mathbf{u}'(t, \mathbf{x})$ is a stochastic field with zero mean. Conditions for representation (2) can be found for example in [8]. From (2) it follows that

$$\dot{\mathbf{Z}} = \mathbf{GZ} + \mathbf{u}'(t, \mathbf{X}(t, \mathbf{a} + \Delta\mathbf{a})) - \mathbf{u}'(t, \mathbf{X}(t, \mathbf{a})).$$

Next assume that the stochastic (turbulent) component $\mathbf{u}'(t, \mathbf{x})$ is homogeneous in space and $\mathbf{u}'(t, \mathbf{x}_1), \quad \mathbf{u}'(t, \mathbf{x}_2)$ are statistically independent whenever $\mathbf{x}_1 \neq \mathbf{x}_2$, i.e. it is a white noise in space. These assumptions imply that statistical characteristics of the Lagrangian separation velocity

$$\mathbf{v}'(t) = \mathbf{u}'(t, \mathbf{X}(t, \mathbf{a} + \Delta\mathbf{a})) - \mathbf{u}'(t, \mathbf{X}(t, \mathbf{a})),$$

depend on time only and we arrive at a Langevin equation

$$\dot{\mathbf{Z}} = \mathbf{GZ} + \mathbf{v}'(t). \tag{3}$$

The main goal of this work is to investigate λ for the simplest hyperbolic structure of the shear tensor

$$\mathbf{G} = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix},$$

implying flow incompressibility, under additional assumption that $\mathbf{v}'(t)$ is a Gaussian white noise with uncorrelated components. In other words we address FSLE for the following Eulerian velocity field

$$u(t, x, y) = \mu x + \sqrt{D}\dot{w}_1, \quad v(t, x, y) = -\mu y + \sqrt{D}\dot{w}_2, \tag{4}$$

with independent white noises, where D is the diffusivity and μ the Lyapunov exponent for the corresponding hyperbolic dynamical system.

Regarding the case of large diffusivity we address an arbitrary constant shear tensor.

Originally, the problem of investigating FSLE was motivated by physical oceanography applications, in particular a question of interest was how effective is FSLE in detecting hyperbolic and other stagnation points of the large scale circulation in the presence of intense small scale turbulence. In view of the strong assumptions leading to (4) that model is hardly adequate to real ocean turbulence in the upper layer. However, it is presumably the only exactly solvable model reflecting both hyperbolic circulation and stochastic perturbations. So, all the effects derived from (4) would be of some interest for further studies in frameworks of more realistic models.

First we briefly discuss the case $D = 0$ for (4), then investigate large and small D, and finally summarize conclusions.

2. Pure dynamics ($D = 0$)

In this case an equation for the separation time τ reads

$$x_0^2 e^{2\mu\tau} + y_0^2 e^{-2\mu\tau} = \alpha^2 (x_0^2 + y_0^2),$$

where (x_0, y_0) is the initial separation. Substituting $x_0 = r_0 \cos\theta, \ y_0 = r_0 \sin\theta$ and solving for τ get

$$\tau = \frac{1}{2\mu} \ln\left(\frac{\alpha^2 \pm \sqrt{\alpha^4 - \sin^2 2\theta}}{2\cos^2\theta}\right).$$

Sign + should be taken otherwise $\tau < 0$. The separation time for the opposite velocity is obtained by replacing θ with $\theta + \pi/2$ Hence

$$T^+ = \frac{1}{2\mu} \ln\left(\frac{\alpha^2 + \sqrt{\alpha^4 - \sin^2 2\theta}}{2\cos^2\theta}\right) \quad T^- = \frac{1}{2\mu} \ln\left(\frac{\alpha^2 + \sqrt{\alpha^4 - \sin^2 2\theta}}{2\sin^2\theta}\right),$$

and FSLE for the unperturbed system

$$\lambda^0 = \ln\alpha\left(\frac{1}{T^+} - \frac{1}{T^-}\right), \tag{5}$$

does not depend on the reference point $\mathbf{a}$ nor on the magnitude of initial separation $|\Delta\mathbf{a}|$ because of translation invariance. Thus it is completely defined by α and the angle θ between $\Delta\mathbf{a}$ and the x-axis. Notice that for horizontal initial separation